# CHARACTERISTIC EQUATION METHOD FOR FRACTAL HEAT-TRANSFER PROBLEM VIA LOCAL FRACTIONAL CALCULUS

by

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Original scientific paper
DOI: 10.2298/TSCI16S3751L

*In this paper the fractal heat-transfer problem described by the theory of local fractional calculus is considered. The non-differentiable-type solution of the heat-transfer equation is obtained. The characteristic equation method is proposed as a powerful technology to illustrate the analytical solution of the partial differential equation in fractal heat transfer.*

Key words: *heat-transfer equation, analytical solution, local fractional calculus, characteristic equation method*

## Introduction

The differential equations involving the local fractional calculus [1] were utilized to investigate the non-differentiable problems, *e. g.*, fractal diffusions [2-7], fractal oscillator [8], fractal wave [9], fractal Laplace [10, 11], fractal heat-conduction [12, 13], fractal Fokker-Planck [14], fractal Helmholtz [15] equations and others [16, 17]. Let us recall the local fractional derivative (LFD) of the function $\Pi(\zeta)$ of order $\theta$ $(0<\theta<1)$ at $\zeta=\zeta_0$, defined by [10-18]:

$$\mathrm{D}_{\zeta}^{(\theta)}\Pi(\zeta_0)=\frac{\mathrm{d}^{\theta}\Pi(\zeta)}{\mathrm{d}\zeta^{\theta}}\Big|_{\zeta=\zeta_0}=\lim_{\zeta\to\zeta_0}\frac{\Delta^{\theta}[\Pi(\zeta)-\Pi(\zeta_0)]}{(\zeta-\zeta_0)^{\theta}} \tag{1}$$

where

$$\Delta^{\theta}[\Pi(\zeta)-\Pi(\zeta_0)]\cong\Gamma(1+\theta)\Delta[\Pi(\zeta)-\Pi(\zeta_0)]$$

The LFD of the function $E_{\theta}(k\zeta^{\theta})(k\in\mathbb{R})$ was [1]:

$$\frac{\mathrm{d}^{\theta}E_{\theta}(k\zeta^{\theta})}{\mathrm{d}\zeta^{\theta}}=kE_{\theta}(k\zeta^{\theta}) \tag{2}$$

The heat-transfer equation involving the LFD in fractal media was written [18]:

$$\frac{\partial^{\theta}\Omega(\vartheta,\tau)}{\partial\tau^{\theta}}+\kappa\frac{\partial^{2\theta}\Omega(\vartheta,\tau)}{\partial\vartheta^{2\theta}}+\omega\Omega(\vartheta,\tau)=0 \tag{3}$$

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where $\kappa$ is a heat-diffusive coefficient and $\omega$ – a constant related to the density and specific heat of fractal materials.

There are a lot of numerical and analytical methods for the local fractional partial differential equations, such as the decomposition method [2, 4, 15], differential transform [3], variational iteration method [5, 12, 14], homotopy perturbation method [6], similarity variable method [7], Laplace variational iteration method [9], series expansion method [10], function decomposition method [11], Fourier transform [13], exp-function method [16], Fourier transform [17], and characteristic equation method (CEM) [19]. The main aim of this paper is to present the CEM to solve the heat-transfer equation in fractal media.

## Solve the heat-transfer equation in fractal media

By using the theory of CEM [19], we set the non-differentiable solution of eq. (3):

$$\Omega(\vartheta, \tau) = E_\theta(\rho\tau^\theta)E_\theta(\sigma\vartheta^\theta) \tag{4}$$

In view of eq. (4), we have:

$$\rho + \kappa\sigma^2 + \omega = 0 \tag{5}$$

such that

$$\Omega(\vartheta, \tau) = \varpi E_\theta[-(\kappa\sigma^2 + \omega)\tau^\theta]E_\theta(\sigma\vartheta^\theta) \tag{6}$$

where $\kappa$ is a heat-diffusive coefficient, $\varpi$ – a constant, and the corresponding graph is represented in fig. 1.

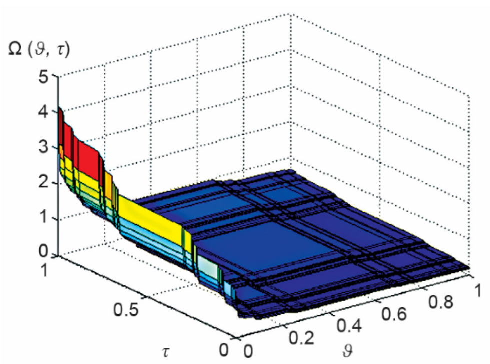


**Figure 1. The solution of non-differentiable type of eq. (5) when $\varpi$ = 1, $\kappa$ = 2, $\omega$ = 1, and $\sigma$ = 1**
*(for color image see journal web-site)*

By changing the dimension from $\theta = \upsilon\ (0 < \upsilon < 1)$ to 1, the conventional heat-transfer equation is written:

$$\frac{\partial\Omega(\vartheta, \tau)}{\partial\tau} + \kappa\frac{\partial^2\Omega(\vartheta, \tau)}{\partial\vartheta^2} + \omega\Omega(\vartheta, \tau) = 0 \tag{7}$$

Then, we obtain:

$$\Omega(\vartheta, \tau) = \varpi \exp[-(\kappa\sigma^2 + \omega)\tau]\exp(\sigma\vartheta) \tag{8}$$

where $\kappa$ is a heat-diffusive coefficient and $\varpi$ – a constant.

Equation (8) represents the heat-transfer equation to account for the radiative loss of heat. The corresponding solutions are illustrated in fig. 2.

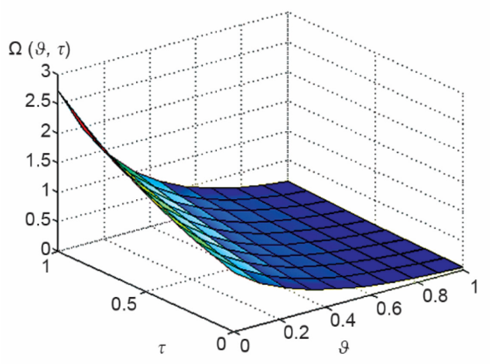


**Figure 2. The differentiable solution for the conventional heat-transfer equation when $\varpi$ = 1, $\kappa$ = 2, $\omega$ = 1, and $\sigma$ = 1**
*(for color image see journal web-site)*

## Conclusion

The fractal heat-transfer problem involving the LFD has been investigated in the work. The non-differentiable solution for the heat-transfer equation in fractal media was obtained by using the CEM. The results for the fractal and conventional heat-transfer equations were compared. The obtained result is very efficient to show the fractal behaviour of heat transfer.

**Acknowledgment**

This work is supported by the Fund for Innovative Research Group of the National Natural Science Foundation of China (Grant No. 51421065), National Natural Science Foundation of China (Grant No. 41471466), Beijing Municipal Natural Science Foundation (Grand No. 8154051), the Priority Development Subject of the Research Fund for the Doctoral Program of Higher Education of China (No. 20110003130003), the Fundamental Research Funds for the Central Universities.

**Nomenclature**

$\theta$ – fractal order, [–]
$\vartheta$ – space co-ordinate, [m]
$\Omega(\vartheta, \tau)$ –temperature, [$Km^{-3}$]
$\tau$ – time, [s]

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Paper submitted: December 5, 2015
Paper revised: January 15, 2016
Paper accepted: January 26, 2016