A LOCAL FRACTIONAL DERIVATIVE WITH APPLICATIONS TO FRACTAL RELAXATION AND DIFFUSION PHENOMENA

by

Yang ZHAO[a], Yan-Guang CAI[a], and Xiao-Jun YANG[b*]

[a] School of Automation, Guangdong University of Technology, Guangzhou, China
[b] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou, China

Original scientific paper
DOI: 10.2298/TSCI16S3723Z

In this paper, a new application of the fractal complex transform via a local fractional derivative is presented. The solution for the fractal relaxation and time-fractal diffusion equations are obtained based on the sup-exponential functions defined on Cantor sets.

Key words: *time-fractal diffusion equation, fractal relaxation, Cantor sets, sup-exponential function*

Introduction

Local fractional derivatives were powerful tools to deal with the real world problems [1-3]. We will now recall them.

The Yang [3] local fractional derivative of the function $\psi(\eta)$ of order $\varepsilon\ (0<\varepsilon<1)$ is defined:

$$D^{(\varepsilon)}\psi(\eta)=\frac{d^{\varepsilon}\psi(\eta)}{d\eta^{\varepsilon}}\Big|_{\eta=\eta_0}=\lim_{\eta\to\eta_0}\frac{\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]}{(\eta-\eta_0)^{\varepsilon}} \tag{1}$$

where

$$\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]\cong\Gamma(1+\varepsilon)\Delta[\psi(\eta)-\psi(\eta_0)] \tag{2}$$

The Chen [4] local fractional derivative of the function $\Phi(\tau)$ of order $\kappa\ (0<\kappa<1)$ is defined:

$$D^{(\kappa)}\Phi(\tau)=\frac{d^{\kappa}\Phi(\tau)}{d\tau^{\kappa}}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Phi(\tau)-\Phi(\tau_0)}{\tau^{\kappa}-\tau_0^{\kappa}} \tag{3}$$

where τ^{κ} is a fractal measure.

The local fractional derivative of the function $\Phi(\tau)$ of order $\kappa(0<\kappa<1)$ is defined by [1]:

$$D^{(\kappa)}\Phi(\tau)=\frac{d^{\kappa}\Phi(\tau)}{d\tau^{\kappa}}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Phi(\tau)-\Phi(\tau_0)}{(\tau-\tau_0)^{\kappa}} \tag{4}$$

where $(\tau-\tau_0)^{\kappa}$ is a fractal measure.

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* Corresponding author; e-mail: dyangxiaojun@163.com

The He [5] local fractional derivative of the function $\Omega(\mu)$ of order $l\ (0<l<1)$ is defined:

$$D^{(l)}\Omega(\mu)=\frac{d\Omega(\mu)}{d\mu^l}\Big|_{\mu=\mu_0}=\frac{d\Omega(\mu)}{d\sigma}=\lim_{\Delta\mu\to\mu_0}\frac{\Omega(\mu_B)-\Omega(\mu_A)}{\Upsilon\eta_0^l} \tag{5}$$

where $d\sigma=\Upsilon\eta_0^l$ with geometrical parameter Υ and measure scale η_0.

The Yang-Dumitru-He local fractional derivative of the function $\Lambda(\eta)$ of order $\zeta\ (0<\zeta<1)$ is defined by [2, 6]:

$$D^{(\varepsilon)}\psi(\eta)=\frac{d^{\varepsilon}\psi(\eta)}{d\eta^{\varepsilon}}\Big|_{\eta=\eta_0}=\lim_{\eta\to\eta_0}\frac{\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]}{(\eta-\eta_0)^{\varepsilon}} \tag{6}$$

where $(\eta-\eta_0)^{\varepsilon}$ represents a fractal measure [3], and:

$$\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]\cong\Gamma(1+\varepsilon)[\psi(\eta)-\psi(\eta_0)] \tag{7}$$

In this paper, we introduce an alternative definition of eq. (6) and a fractal complex transform, which is alternative transformation [6, 7].

A local fractional derivative and its properties

The local fractional derivative of the function $\Lambda(\eta)$ of order $\zeta(0<\zeta<1)$ is defined by [8]:

$$D^{(\zeta)}\Lambda(\eta)=\frac{d^{\zeta}\Lambda(\eta)}{d\eta^{\zeta}}\Big|_{\eta=\eta_0}=\lim_{\eta\to\eta_0}\frac{\Lambda(\eta)-\Lambda(\eta_0)}{(\eta-\eta_0)^{\zeta}} \tag{8}$$

where $(\eta-\eta_0)^{\zeta}$ represents a fractal measure, see the graph [3].

In fact, eq. (5) is alternative definition of the local fractional derivative [5] after cancelling the gamma function.

Properties of the local fractional derivative operator are given [8]:

(M1) $$D^{(\zeta)}[\Lambda_1(\eta)+\Lambda_2(\eta)]=D^{(\zeta)}\Lambda_1(\eta)+D^{(\zeta)}\Lambda_2(\eta)$$

(M2) $$D^{(\zeta)}\Lambda[\mu(\eta)]=D_{\mu}^{(1)}\Lambda[\mu(\eta)]D^{(\zeta)}\mu(\eta)$$

(M3) $$D^{(\zeta)}[\Lambda_1(\eta)/\Lambda_2(\eta)]=[\Lambda_2(\eta)D^{(\zeta)}\Lambda_1(\eta)-\Lambda_1(\eta)D^{(\zeta)}\Lambda_2(\eta)]/[\Lambda_2(\eta)]^2$$

(M4) $$D^{(\zeta)}H=0$$

(M5) $$D^{(\zeta)}\eta^{\zeta}=1$$

(M6) $$D^{(\zeta)}\sup\exp(\eta^{\zeta})=\sup\exp(\eta^{\zeta})$$

where $\Lambda_2(\eta)\neq 0$, H is a constant, η^{ζ} is a fractal measure, and $\sup\exp(\eta^{\zeta})=\Sigma_{i=0}^{\infty}\eta^{\zeta i}/\Gamma(i+1)$ is a sup-exponential functions defined on Cantor sets. The proofs of them are similar to the processes [3].

Two examples

Example 1

As a simple application, we consider the fractal relaxation with the local fractional derivative:

$$\mathrm{D}^{(\zeta)}\Lambda(\eta)+\varpi\Lambda(\eta)=0, \qquad 0<\zeta<1, \qquad \Lambda(\eta)=1 \tag{9}$$

With the help of the transformation [8], which is given by $\mathrm{D}^{(\zeta)}\Lambda[\mu(\eta)]= = \mathrm{D}_{\mu}^{(1)}\Lambda[\mu(\eta)]\mathrm{D}^{(\zeta)}\eta^{\zeta}$, where $\mu(\eta)=\eta^{\zeta}$, eq. (8) can be equivalently written in the form:

$$\mathrm{D}^{(1)}\Lambda(\mu)+\varpi\Lambda(\mu)=0 \tag{10}$$

The solution of eq. (9) is:

$$\Lambda(\mu)=\exp(-\varpi\mu) \tag{11}$$

Making use of $\mu(\eta)=\eta^{\zeta}$, we obtain the solution of eq. (8), namely:

$$\Lambda(\eta)=\sup\exp(-\varpi\eta^{\zeta}) \tag{12}$$

and its corresponding graph for the fractal dimension $\zeta=\ln 2/\ln 3$ and $\varpi=1$ is represented in fig. 1.

The comparative result between fractal relaxation phenomena with the local fractional derivatives eqs. (1) and (7) is shown in fig. 2.

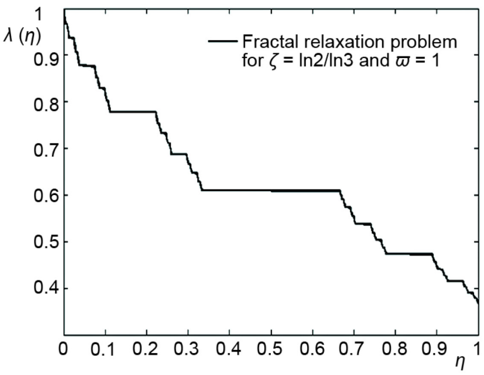

Figure 1. The non-differentiable solution of the fractal relaxation problem for ζ = ln2/ln3 and ϖ = 1

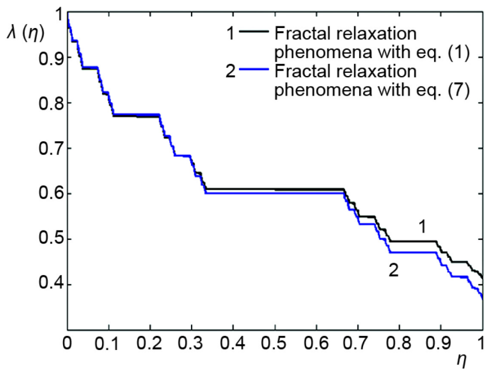

Figure 2. The comparative result between fractal relaxation phenomena in the different operators

The transformation, which is reduced to by using the expression (M2), is called as the fractal complex transform.

Example 2

In this example, we consider the fractal-time diffusion equation with the local fractional derivative:

$$\frac{\partial^{\zeta}\Lambda(\eta,\tau)}{\partial\tau^{\zeta}}=\frac{\partial^{2}\Lambda(\eta,\tau)}{\partial\eta^{2}}, \qquad 0<\zeta<1, \qquad \Lambda(\eta,0)=\delta(\eta) \tag{13}$$

By using the fractal complex transform:

$$\frac{\partial^{\zeta}\Lambda[\eta,\mu(\tau)]}{\partial\tau^{\zeta}}=\frac{\partial\Lambda[\eta,\mu(\tau)]}{\partial\mu}\mathrm{D}^{(\zeta)}\tau^{\zeta}$$

where $\mu(\tau)=\tau^{\zeta}$, we have:

$$\frac{\partial\Lambda(\mu,\mu)}{\partial\mu}=\frac{\partial^{2}\Lambda(\eta,\mu)}{\partial\eta^{2}},\qquad \Lambda(\eta,0)=\delta(\eta) \tag{14}$$

The solution of eq. (13) reads [9]:

$$\Lambda(\mu,\mu)=\frac{1}{\sqrt{4\pi\mu}}\exp\left(-\frac{\mu^{2}}{4\mu}\right) \tag{15}$$

and its picture is shown in fig. 3.

Thus, we present the non-differentiable solution of eq. (13), namely:

$$\Lambda(\eta,\tau)=\frac{1}{\sqrt{4\pi\tau^{\zeta}}}\sup\exp\left(-\frac{\mu^{2}}{4\tau^{\zeta}}\right) \tag{16}$$

and its graph is illustrated in fig. 4.

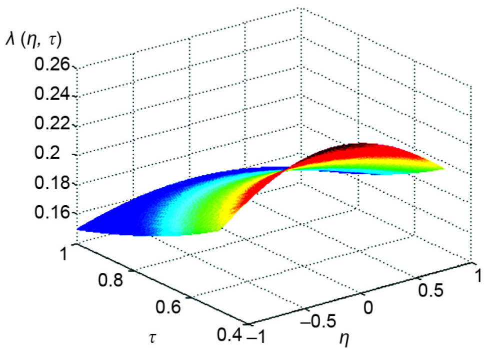

Figure 3. The picture of eq. (14)
(for color image see journal web-site)

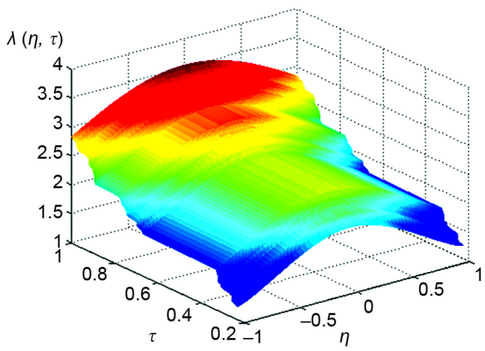

Figure 4. The non-differentiable solution for ζ = ln2/ln3 *(for color image see journal web-site)*

Conclusions

In this work, we obtained the fractal complex transform method involving local fractional derivative. We used the new technology to solve the fractal relaxation and diffusion phenomena. The results are easily obtained by the proposed technology.

Nomenclature

ζ – fractal dimension, [–]
η – space co-ordinate, [m]
$\Lambda(\eta, \tau)$ – concentration, [–]
τ – time, [s]

References

[1] Yang, X. J., *Local Fractional Functional Analysis & Its Applications,* Hong Kong: Asian Academic Publisher Limited, Hong Kong, China, 2011
[2] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications*, World Science, New York, NY, USA, 2012
[3] Yang, X. J. *et al.*, *Local Fractional Integral Transforms and Their Applications*, Academic Press, New York, USA, 2015
[4] Chen, W., Time-space Fabric Underlying Anomalous Diffusion, *Chaos, Solitons & Fractals*, *28* (2006), 4, pp. 923-929
[5] He, J.-H., A New Fractal Derivation, *Thermal Science*, *15* (2011), Suppl. 1, pp. S145-S147
[6] Yang, X. J., *et al.*, Transport Equations in Fractal Porous Media within Fractional Complex Transform Method, *Proceedings,* Romanian Academy, Series A, 2013, Vol. 14, No. 4, pp. 287-292
[7] Su, W. H., *et al.*, Fractional Complex Transform Method for Wave Equations on Cantor Sets within Local Fractional Differential Operator, *Advances in Difference Equations*, *2013* (2013), 1, pp. 1-8
[8] Yang, X. J., *et al.*, A New Local Fractional Derivative and Its Application to Diffusion in Fractal Media, *Computational Method for Complex Science, 1* (2016), 1, pp. 1-5
[9] Crank, J., *The Mathematics of Diffusion,* Oxford University Press, Oxford, UK, 1979

Paper submitted: February 1, 2016
Paper revised: March 10, 2016
Paper accepted: March 26, 2016