SOLVING NON-LOCAL FRACTICAL HEAT EQUATIONS BASED ON THE REPRODUCING KERNEL METHOD

by

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In this paper, a numerical method is proposed for 1-D fractional heat equations subject to non-local boundary conditions. The reproducing kernel satisfying non-local conditions is constructed and reproducing kernel theory is applied to solve the considered problem. A numerical example is given to show the effectiveness of the method.

Key words: *reproducing kernel method, non-local boundary conditions, fractional, heat equation,*

Introduction

Non-local conditions arise mainly when the data on the boundary may not be measured directly. Boundary value problems with non-local conditions have important application [1, 2]. Recently, the fractional heat equations with non-local boundary conditions have received much attention. Fractional problems have been studied by many authors [3-13]. There is much discussion about fractional heat equations with local boundary conditions. The theoretical aspects of the solutions to the problems have been discussed by some authors [6]. Molliq *et al.* [7] applied the variational iteration method to obtain analytical solutions of fractional heat- and wave-like equations with variable coefficients. Scherer *et al.* [8] considered the numerical treatment of some fractional extensions of the temperature field problem in oil strata. Based on Legendre polynomials, Khalil and Khan [9] developed a new scheme for numerical solutions of the fractional 2-D heat conduction equations on a rectangular plane. Sarwar *et al.* [10] discussed the optimal homotopy asymptotic method for finding the optimal solutions of fractional order heat- and wave-like equations. Xu *et al.* [11], Zhang *et al.* [12], and Zhao *et al.* [13] also discussed some numerical methods for fractional heat equations. The 1-D classical heat equation with the non-local condition is solved in [14-16]. However, the numerical discussion on fractional heat equations with non-local boundary conditions is rare. Karatay *et al.* [17] proposed a method for solving inhomogeneous non-local fractional heat equations based on the modified Gauss elimination method.

Consider the following non-local fractional heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^{\alpha} u}{\partial x^{\alpha}}+a(x,t)u(x,t)+g(x,t), \quad 0<x<X, \ 0<t\le T \tag{1}$$

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subject to the initial condition:

$$u(x,0)=f(x) \tag{2}$$

and the non-local boundary conditions:

$$u(0,t)+\mu_1 u(\xi,t)=\int_0^X h_1(x)u(x,t)\mathrm{d}x, \qquad u(X,t)+\mu_2 u(\eta,t)=\int_0^X h_2(x)u(x,t)\mathrm{d}x \tag{3}$$

where $1<\alpha<2$, $0<\xi,\ \eta<X,$ and the fractional derivative in the Caputo sense is given by:

$$\frac{\partial^\alpha u}{\partial x^\alpha}=\begin{cases}\dfrac{1}{\Gamma(2-\alpha)}\displaystyle\int_0^x (x-s)^{1-\alpha}\frac{\partial^2 u(s,t)}{\partial s^2}\mathrm{d}s, & 1<\alpha<2\\[2ex] \dfrac{\partial^2 u(x,t)}{\partial x^2}, & \alpha=2\end{cases} \tag{4}$$

The reproducing kernel method for solving operator equation has been proposed and applied to many fields [18-28]. In this paper, we shall introduce the numerical method for obtaining the approximate solution of problem in eqs. (1)-(3) based on the transverse method of lines and the reproducing kernel method.

Method for problem (1)-(3)

To solve problem (1)-(3), we discretize the time variable first. Divide $[0, T]$ into m equal subintervals $[t_{j-1}, t_j]$, $j = 1, 2,\ldots, m$, with $t_i = i\Delta t_i$ and $\Delta t = T/m$. Denote $u(x, t_i)$ and $a(x, t_i)$ by $u_i(x)$ and $a_i(x)$, respectively. By approximating the time derivative $\partial u/\partial t$ by a backward difference approximation, problem in eqs. (1)-(3) is reduced to:

$$\begin{cases}\dfrac{\partial^\alpha u_i}{\partial x^\alpha}+[a_i(x)-\Delta t]u_i(x)=-g(x,t_i)-\Delta t u_{i-1}(x)\triangleq G_i(x), & 0<x<X\\[2ex] u_i(0)+\mu_1 u_i(\xi)=\displaystyle\int_0^X h_1(x)u_i(x)\mathrm{d}x, \qquad u_i(X)+\mu_2 u_i(\eta)=\int_0^X h_2(x)u_i(x)\mathrm{d}x\end{cases} \tag{5}$$

Clearly, problems in eq. (5) are fractional non-local ordinary differential equation boundary value problems for space variable x. We will introduce the reproducing kernel method for solving eq. (5).

To illustrate the method for solving eq. (5), we consider the following problem:

$$\begin{cases}\dfrac{\partial^\alpha u_i}{\partial x^\alpha}+a(x)u(x)=\bar{F}(x), & 0<x<X\\[2ex] u(0)+\mu_1 u(\xi)=\displaystyle\int_0^X h_1(x)u(x)\mathrm{d}x, \qquad u(X)+\mu_2 u(\eta)+\int_0^X h_2(x)u(x)\mathrm{d}x\end{cases} \tag{6}$$

where $a(x)$ and $\bar{F}(x)$ are given continuous functions.

Applying Riemann-Liouville fractional integral operator I^α to both sides of eq. (6), we have:

$$\begin{cases} u(x)-u(0)-u'(0)x+I^{\alpha}[a(x)u(x)]=F(x), & 0<x<X \\ u(0)+\mu_1 u(\xi)=\int_0^X h_1(x)u(x)\mathrm{d}x, & u(X)+\mu_2 u(\eta)=\int_0^X h_2(x)u(x)\mathrm{d}x \end{cases} \tag{7}$$

where $F(x)=I^{\alpha}\bar{F}(x)$.

To solve eq. (7) based on the reproducing kernel theory, we first construct a reproducing kernel space $W^3[0, X]$ in which every function satisfies the non-local boundary conditions of eq. (7).

Definition 1. $W^3[0, X] = \{u(x)|u''(x)$ is an absolutely continuous real value function:

$$u'''(x)\in L^2[0,X], \qquad u(0)+\mu_1 u(\xi)=\int_0^X h_1(x)u(x)\mathrm{d}x, \qquad u(X)+\mu_2 u(\eta)=\int_0^X h_2(x)u(x)\mathrm{d}x \Big\}$$

equipped with the following inner product and norm:

$$[u(y),v(y)]_{W^3}=u(0)v(0)+u'(0)v'(0)=u(X)v(X)+\int_0^X u'''v'''\mathrm{d}y$$

and

$$\|u\|_{W^3}=[u(y),v(y)]_{W^3}, \qquad u, \qquad v\in W^3[0,X]$$

From the results in [18], it is easy to prove that $W^3[0, X]$ is a reproducing kernel space. For the method for constructing its reproducing kernel $k(x, y)$, please refer to [24].

Definition 2. $W^1[0, X] = \{u(x)|u(x)$ is an absolutely continuous real value function, $u'(x) \in L^2[0, X]\}$ equipped with the following inner product and norm:

$$[u(y),v(y)]_{W^1}=u(0)v(0)+\int_0^X u'v'\mathrm{d}y$$

and

$$\|u\|_{W^1}=[u(y),v(y)]_{W^1}, \quad u, \quad v\in W^1[0,X]$$

Its reproducing kernel $k(x, y)$ can also be obtained easily.

Define linear operator $Lu(x) = u(x) - u(0) - u'(0)x + I^{\alpha}[a(x)u(x)]$. It is easy to show that operator $L: W^3[0, X] \rightarrow W^1[0, X]$ is bounded. Put $\varphi_i(x) = \bar{k}(x, x_i)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. Performing Gram-Schmidt orthogonalization to $\{\psi_i(x)\}_{i=1}^{\infty}$, we can get an orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$:

$$\bar{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x), \qquad \beta_{ii}>0, \quad i=1,2,...$$

From [18], we have the following theorem:

Theorem 1. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, X]$, then $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^3[0, X]$ and $\psi_i(x)$ can be represented explicitly by $\psi_i(x) = L_y k(x, y)|_{y=x_i}$.

Theorem 2. Under the condition of Theorem 1, the solution of eq. (7) can be represented by:

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_k(x) \tag{8}$$

Proof. From Theorem 1, it follows that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is a complete orthonormal basis of $W^3[0, X]$. Furthermore, we have:

$$\begin{aligned}u(x)&=\sum_{i=1}^{\infty}[u(x),\quad \overline{\psi}_i(x)]\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[u(x),\quad L^{*}\varphi_k(x)]\overline{\psi}_k(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[Lu(x),\quad \varphi_k(x)]\overline{\psi}_k(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[F(x),\quad \varphi_k(x)]\overline{\psi}_k(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_k(x)\end{aligned}$$

and the proof of the theorem is complete.

An approximation to solution of eq. (7) can be obtained naturally by taking the N-term intercept of $u(x)$:

$$u_N(x)=\sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_k(x)$$

By using the previousmethod, one can obtain the approximate solutions of problem in eq. (5):

$$u_{j,N}(x)=\sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}G_j(x_k)\overline{\psi}_k(x),\qquad j=1,2,...,m$$

Combination of all $u_{j,N}(x)$, $1 \le j \le m$, leads to the approximation to the solution of eqs. (1)-(3).

A typical example

Consider the following non-local fractional heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^{1.3}u}{\partial x^{1.3}}=u(x,t)=(2e^{-t}-1)(x^2-x^4)-\frac{\Gamma(3)}{\Gamma(1.7)}x^{0.7}-\frac{\Gamma(5)}{\Gamma(3.7)}x^{2.7},\quad 0<x<1,\quad 0<t\le 1$$

with an initial condition $u(x, 0) = 0$, and two non-local boundary conditions:

$$u(0,t)-u\left(\frac{1}{4},t\right)=-\frac{225}{512}\int_0^1 u(x,t)\mathrm{d}x,\qquad u(1,t)-u\left(\frac{1}{2},t\right)=-\frac{45}{32}\int_0^1 u(x,t)\mathrm{d}x$$

The exact solution is $u(x, t) = (x^2 - x^4)(1 - \mathrm{e}^{-t})$.

Taking $N = 10$, $m = 20, 40$, $x_i = (i - 1)/(N - 1)$, $(i = 1, 2,\ldots, N)$, the absolute errors of approximate solution obtained by the present method are shown in figs. 1 and 2.

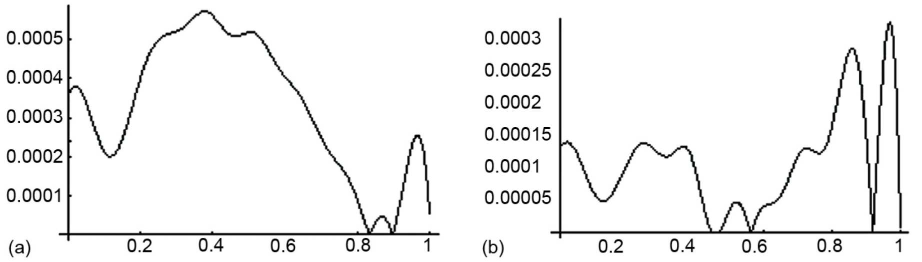

Figure 1. Absolute errors for *m* = 20; (a) *t* = 0.5, (b) *t* = 1

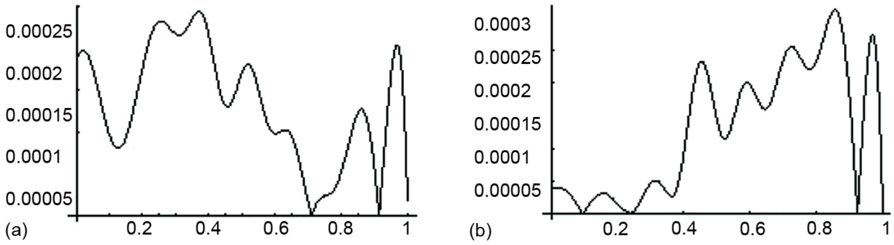

Figure 2. Absolute errors for *m* = 20; (a) *t* = 0.5, (b) *t* = 1

Conclusions

In this paper, by combining the reproducing kernel method for solving non-local fractional ordinary differential equations and the transverse method of lines, an effective numerical method is proposed for non-local fractional heat equations. The numerical results show that the present method can provide good numerical approximation and is a reliable technique.

Nomenclature

a – diffusion coefficient, [m^2s^{-1}]
m – integer, [–]
N – integer, [–]
t – time, [s]
u – temperature, [K]
x – displacement, [m]

Greek symbol

α – fractional order, [–]

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