A NOVEL SERIES METHOD FOR FRACTIONAL DIFFUSION EQUATION WITHIN CAPUTO FRACTIONAL DERIVATIVE

by

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In this paper, we suggest the series expansion method for finding the series solution for the time-fractional diffusion equation involving Caputo fractional derivative.

Key words: *time-fractional diffusion equation, series expansion method, series solution, Caputo fractional derivative*

Introduction

The fractional derivatives [1-6] were the potential tools for modelling the complex behaviors in science and engineering. Mathematical theory of time-fractional diffusion equation within Caputo fractional derivative:

$$\frac{\partial^{\gamma}\Omega(x,t)}{\partial t^{\gamma}} = \mu \frac{\partial^{2}\Omega(x,t)}{\partial x^{2}} \tag{1}$$

was studied in [7], where μ is a constant. The Caputo fractional derivative of the continuous function $\phi(x)$ of fractional order γ is defined as [3, 6]:

$$\mathrm{D}_{a^+}^{(\gamma)}\phi(x) = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{1}{(x-\tau)^{\gamma}}\frac{\mathrm{d}\phi(\tau)}{\mathrm{d}x}\,\mathrm{d}\tau \tag{2}$$

where $0<\gamma<1$ and $x > a$.

The prosperities of the Caputo fractional derivatives of the continuous functions are given as [3, 6]:

$$\mathrm{D}_{a^+}^{(\gamma)}1 = \frac{(x-a)^{-\gamma}}{\Gamma(1-\gamma)} \tag{3a}$$

$$\mathrm{D}_{a^+}^{(\gamma)}[(x-a)^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(x-a)^{\beta-\gamma} \tag{3b}$$

$$\mathrm{D}_{a^+}^{(\gamma)}\{E_{\gamma}[(x-a)^{\gamma}]\} = E_{\gamma}[(x-a)^{\gamma}] \tag{3c}$$

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where the generalized Mittag-Leffler function is defined as [3, 6]:

$$E_\gamma(x^\gamma) = \sum_{i=0}^{\infty} \frac{x^{i\gamma}}{\Gamma(1+i\gamma)} \tag{3d}$$

The fractional diffusion problems were solved by the different technologies, such as the Chebyshev pseudospectral method [8], space-time spectral method [9], ractional-order implicit difference method [10], legendre spectral element method [11], homotopy perturbation method [12], differential transform method [13], similarity variable method [14], and Laplace series expansion method [15].

The local fractional series expansion method was proposed to solve the diffusion [16] and Schroedinger [17] equations defined on Cantor sets. Its extended version (fractional series expansion method) via modified Riemann-Liouville derivative was discussed in [18]. However, the series expansion method for handling the fractional differential equations involving the Caputo fractional derivative have not proposed. The main aim of the paper is to propose the series expansion method to solve the time-fractional diffusion equation with the Caputo fractional derivative.

Analysis of the method

In order to introduce the technology, we now give the time-fractional diffusion equation:

$$\Omega_\tau^{(\gamma)} = L\Omega \tag{4}$$

where $L = \mu\partial^2/\partial x^2$ is a linear operator with respect to *x*.

We introduce a multi-term separated functions of independent variables *t* and *x*, namely:

$$\Omega(x, t) = \sum_{i=0}^{\infty} \mathrm{M}_i(t)\Psi_i(x) \tag{5}$$

where $\mathrm{M}_i(t)$ and $\Psi_i(x)$ are the continuous functions.

Define the series term:

$$\mathrm{M}_i(t) = \nu_i \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \tag{6}$$

where ν_i is a coefficient.

Substituting eq. (6) into eq. (5) gives:

$$\Omega(x, t) = \sum_{i=0}^{\infty} \nu_i \frac{t^{i\gamma}}{\Gamma(1+i\gamma)}\Psi_i(x) \tag{7}$$

By using $\nu_i = 1$, from eq. (7) we obtain:

$$\Omega(x, t) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)}\Psi_i(x) \tag{8}$$

Submitting eq. (8) into eq. (4), we have:

$$\Omega_{\tau}^{(\gamma)} = \sum_{i=1}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} X_i(x) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \Psi_{i+1}(x) \tag{9}$$

and

$$L\Omega = L\left[\sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \Psi_i(x)\right] = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} (L\Psi_i)(x) \tag{10}$$

From eqs. (9), (10), and (4), we get:

$$\sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \Psi_{i+1}(x) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} (L\Psi_i)(x) \tag{11}$$

which leads to the recursion:

$$\Psi_{i+1}(x) = (L\Psi_i)(x) \tag{12}$$

With the help of eq. (12), we obtain the series solution of eq. (4), namely:

$$\Omega(x, t) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \Psi_i(x) \tag{13}$$

where the convergent condition is:

$$\lim_{n\to\infty}\left[\frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \Psi_i(x)\right] = 0 \tag{14}$$

The technology is called as the series expansion method.

Solving time-fractional diffusion equation

We consider the following initial value of eq. (1) given as:

$$\Omega(x, 0) = \mathrm{e}^x \tag{15}$$

With the help of eqs. (12) and (14), we can structure the following iterative formula:

$$\begin{cases} \Psi_{i+1}(x) = \left(\mu \dfrac{\mathrm{d}^2\Psi_i}{\mathrm{d}x^2}\right)(x) \\ \Psi_0(x) = \Omega(x, 0) = \mathrm{e}^x \end{cases} \tag{16}$$

which leads to the following terms:

$$\Psi_1(x) = \mu \mathrm{e}^x, \qquad \Psi_2(x) = \mu^2 \mathrm{e}^x, \qquad \ldots \qquad , \Psi_i(x) = \mu^i \mathrm{e}^x$$

Therefore, we obtain the series solution:

$$\Omega(x, t) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}}{\Gamma(1+i\gamma)} \mu^i \mathrm{e}^x = \mathrm{e}^x \sum_{i=0}^{\infty} \frac{t^{i\gamma}\mu^i}{\Gamma(1+i\gamma)} = \mathrm{e}^x E_\gamma(\mu t^\gamma) \tag{17}$$

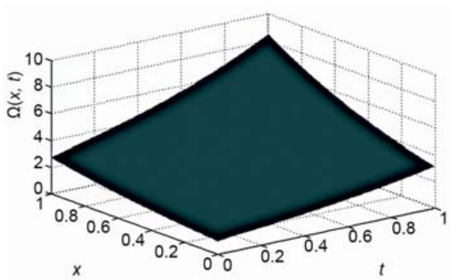

Figure 1. The solution of eq. (1) based on the generalized Mittag-Leffler function for $\mu = 1$ and $\gamma = 0.85$

where the generalized Mittag-Leffler function is defined as [3, 6]:

$$E_\gamma(\mu t^\gamma) = \sum_{i=0}^{\infty} \frac{t^{i\gamma}\mu^i}{\Gamma(1+i\gamma)} \tag{18}$$

The graph of the solution of eq. (1) is shown in fig. (1).

Conclusions

In this work, we presented the new method for solving the time-fractional diffusion equation involving Caputo fractional derivative. The series expansion terms are based on the generalized Mittag-Leffler function. This technology is accurate and efficient for the fractional differential equations.

Nomenclature

x – space co-ordinate, [m]
t – time, [s]

Greek symbols

γ – fractional order, [–]
$\Omega(x, t)$ – concentration, [–]

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