EXACT SOLUTIONS OF TIME FRACTIONAL HEAT-LIKE AND WAVE-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS

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In this paper, a variable-coefficient time fractional heat-like and wave-like equation with initial and boundary conditions is solved by the use of variable separation method and the properties of Mittag-Leffler function. As a result, exact solutions are obtained, from which some known special solutions are recovered. It is shown that the variable separation method can also be used to solve some others time fractional heat-like and wave-like equation in science and engineering.

Key words: *heat-like and wave-like equation, variable separation method, exact solution, Mittag-Leffler function*

Introduction

As the generalizations of classical differential equations with integer order, fractional differential equations (FDE) have attached much attention [1-7]. With the development of fractional calculus theory, solving FDE has become one of the most important and significant research tasks in science and engineering. However, as pointed in [8] that it is still on a preliminary stage to search for exact analytical solutions of non-linear FDE. Recently, Zhang *et al.* [9] obtained many exact solutions of a (2+1)-D non-linear time fractional biological population model by means of variable separation method. In general, there are three steps in the variable separation method for FDE with time fractional derivative, say, in four variables *x, y, z,* and *t.* Step 1: write one solution u of the FDE as a product of two undetermined functions $v(t)$ and $w(x, y, z)$. Step 2: substitute $u = v(t)w(x, y, z)$ into the given FDE and then convert the resulting equation into two differential equations, one of which is a solvable FDE with time fractional derivative and the other is a solvable partial differential equation (PDE) depending only on *x, y,* and *z*. Step 3: solve the reduced FDE and PDE, respectively, and hence obtain some variable separation solutions of the original FDE.

When the inhomogeneities of media and non-uniformities of boundaries are taken into account, the variable-coefficient equations could describe more realistic physical phenomena than their constant-coefficient counterparts [10]. Therefore, how to construct exact solutions of FDE with variable coefficients is worth studying. The present paper is motivated by the desire to extend the variable separation method to the following variable-coefficient time fractional heat-like and wave-like equations [11, 12]:

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$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz} + \theta(x, y, z),$$
$$0 < x < a, \qquad 0 < y < b, \qquad 0 < z < c, \qquad t > 0 \tag{1}$$

subject to boundary conditions:

$$u_x(0, y, z, t) = f_1(y, z, t), \qquad u_x(a, y, z, t) = f_2(y, z, t) \tag{2}$$

$$u_y(x, 0, z, t) = g_1(x, z, t), \qquad u_y(x, b, z, t) = g_2(x, z, t) \tag{3}$$

$$u_z(x, y, 0, t) = h_1(x, y, t), \qquad u_z(x, y, c, t) = h_2(x, y, t) \tag{4}$$

and the initial conditions:

$$u(x, y, z, 0) = \eta(x, y, z), \qquad u_t(x, y, z, 0) = \varphi(x, y, z) \tag{5}$$

where α is a parameter describing the fractional derivative, u_t is the rate of change of temperature at a point over time. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $0 < \alpha \le 1$, eq. (1) reduces to the fractional heat-like equation with variable coefficients, and it does to a wave-like equation with variable coefficients for $1 < \alpha \le 2$.

Exact solutions

In this section, we use the variable separation method previously described to solve eq. (1) with the fractional derivative in the Caputo sense [11]:

$$\frac{\partial^\alpha u(x, y, z, t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x, y, z, \tau)}{\partial \tau^m} \mathrm{d}\tau \tag{6}$$

for $m - 1 < \alpha < m$, $m \in \mathbb{N}$, $x > 0$. Some properties of the Caputo time-fractional derivative are used:

$$\mathrm{D}_t^\alpha [\lambda f(t) + \mu g(t)] = \lambda \mathrm{D}_t^\alpha f(t) + \mu \mathrm{D}_t^\alpha g(t) \tag{7}$$

$$\mathrm{D}_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \qquad \gamma > 0 \tag{8}$$

$$\mathrm{D}_t^\alpha E_\alpha(qt^\alpha) = qE_\alpha(t^\alpha) \tag{9}$$

where λ, μ, and q are constants or functions independent of t, $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \tag{10}$$

To begin with, we take a transformation:

$$u = v(t)w, \qquad w = w(x, y, z) \tag{11}$$

then eq. (1) is converted into:

$$(\mathrm{D}_t^\alpha v)w = (fw_{xx} + gw_{yy} + hw_{zz})v + \theta \tag{12}$$

Further supposing that:

$$v = \vartheta - \frac{\theta}{fw_{xx} + gw_{yy} + hw_{zz}}, \qquad \vartheta = \vartheta(x, y, z) \tag{13}$$

and substituting eq. (13) into eq. (12), we have:

$$\mathrm{D}_t^\alpha \vartheta = \frac{fw_{xx} + gw_{yy} + hw_{zz}}{w} \vartheta \tag{14}$$

Solving eq. (14) with the help of eq. (9), we obtain:

$$\vartheta = E_\alpha \left(\frac{fw_{xx} + gw_{yy} + hw_{zz}}{w} t^\alpha \right) \tag{15}$$

and hence have

$$u = E_\alpha \left(\frac{fw_{xx} + gw_{yy} + hw_{zz}}{w} t^\alpha \right) w - \frac{\theta w}{fw_{xx} + gw_{yy} + hw_{zz}} \tag{16}$$

Obviously, if eq. (16) is a solution of eq. (1), then it must satisfiy eqs. (2)-(5).

Examples

In this section, we further determine solution (16) through four examples.

Example 1: we consider the 1-D fractional heat-like problem of form:

$$\mathrm{D}_t^\alpha u = \frac{1}{2} x^2 u_{xx}, \qquad 0 < x < 1, \qquad 0 < \alpha \le 1, \qquad t > 0 \tag{17}$$

subject to the boundary conditions:

$$u(0, t) = 0, \qquad u(1, t) = E_\alpha(t^\alpha) \tag{18}$$

and the initial condition:

$$u(x, 0) = x^2 \tag{19}$$

In this case of example 1, we can easy to see that $f = x^2/2$, $g = h = \theta = 0$, and $w = x^2$. Thus, we have:

$$u(x, t) = E_\alpha(t^\alpha) x^2 \tag{20}$$

When we set $\alpha = 1$, solution (20) becomes the known one $u(x, t) = x^2 e^t$, which was found in [12].

Example 3: We consider in this example the 2-D fractional heat-like equation:

$$\mathrm{D}_t^\alpha u = u_{xx} + u_{yy}, \qquad 0 < x, y < 2\pi, \qquad 0 < \alpha \le 1, \qquad t > 0 \tag{21}$$

subject to the boundary conditions:

$$u(0, y, t) = 0, \qquad u(2\pi, y, t) = 0 \tag{22}$$

$$u(x, 0, t) = 0, \qquad u(x, 2\pi, t) = 0 \tag{23}$$

with the initial condition:

$$u(x, y, 0) = \sin x \sin y \tag{24}$$

It can be easily seen from this example that $f = g = 1,$ $h = \theta = 0,$ and $w = \sin x \sin y.$ We therefore obtain:

$$u(x, y, t) = E_\alpha(-2t^\alpha) \sin x \sin y \tag{25}$$

For the special case $\alpha = 1,$ the known solution [12] $u(x, t) = e^{-2t} \sin x \sin y$ can be recovered.

Example 4: In this example, we consider the 3-D fractional wave-like equation in the form:

$$\mathrm{D}_t^\alpha u = x^2 + y^2 + z^2 + \frac{1}{2}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}), \quad 0 < x, y, z < 1, \quad 1 < \alpha \le 2, \quad t > 0 \tag{26}$$

subject to the boundary conditions:

$$u(0, y, z, t) = (y^2 + z^2)[E_\alpha(t^\alpha) - 1] \tag{27}$$

$$u(1, y, z, t) = (1 + y^2 + z^2)[E_\alpha(t^\alpha) - 1] \tag{28}$$

$$u(x, 0, z, t) = (x^2 + z^2)[E_\alpha(t^\alpha) - 1] \tag{29}$$

$$u(x, 1, z, t) = (1 + x^2 + z^2)[E_\alpha(t^\alpha) - 1] \tag{30}$$

$$u(x, y, 0, t) = (x^2 + y^2)[E_\alpha(t^\alpha) - 1] \tag{31}$$

$$u(x, y, 1, t) = (1 + x^2 + y^2)[E_\alpha(t^\alpha) - 1] \tag{32}$$

with the initial condition:

$$u(x, y, z, 0) = 0, \qquad u_t(x, y, z, 0) = x^2 + y^2 + z^2 \tag{33}$$

We can see from this case of example 4 that $f = x^2/2,$ $g = y^2/2,$ $h = z^2/2,$ and $w = \theta = x^2 + y^2 + z^2.$ We then obtain:

$$u(x, y, z, t) = (x^2 + y^2 + z^2)[E_\alpha(t^\alpha) - 1] \tag{34}$$

If let $\alpha = 2,$ then eq. (34) gives a solution in the closed form:

$$u(x, y, z, t) = (x^2 + y^2 + z^2) \cosh t \tag{35}$$

Nomenclature

a – real number, [m]
b – real number, [m]
c – real number, [m]
D_t^α – Caputo fractional derivative, [–]
$E_\alpha(t^\alpha)$ – Mittag-Leffler function, [–]
k – integer, [–]
m – integer, [–]
q – constant, [–]
t – time, [s]
x, y, z – displacements, [m]

Greek symbols

α – order of fractional derivative, [–]
γ – real number, [–]
λ – constant, [–]
μ – constant, [–]
π – circumference ratio, [–]
τ – time, [s]
Γ – gamma function, [–]
$\mathbb{N}$ – integer set, [–]

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