APPROXIMATE SOLUTION OF THE NON-LINEAR DIFFUSION EQUATION OF MULTIPLE ORDERS

by

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In this paper, fractional diffusion equation of multiple orders is approximately solved. The equation is given in the equivalent integral form. The Adomian polynomial is adopted and analytical solutions are obtained. The result contains two parameters that can have more space for fitting the experiment data.

Key words: *new Adomian polynomials, approximate solution, fractional calculus, non-linear diffusion equation*

Introduction

Fractional derivative has a feature of non-locality property. Besides, it also holds other analytical properties such as the Leibniz integral law, *etc.* Particularly, for the diffusion issue, the procedure often exhibits randomicity and history dependence. As a result, take the normal non-linear diffusion equation for example:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + g(u) \tag{1}$$

where D is the diffusion coefficient and $g(u)$ is a non-linear term with respect to u.

Considering the memory effects, it can be modified:

$${}_a^C\mathrm{D}_t^\alpha u = D\frac{\partial^2 u}{\partial x^2} + g(u), \qquad 0<\alpha<1 \tag{2}$$

where ${}_a^C\mathrm{D}_t^\alpha$ is the left Caputo derivative [1], and it is defined:

$${}_0^C\mathrm{D}_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha} u'(\tau)\mathrm{d}\tau, \qquad 0<\alpha<1, \qquad 0<t \tag{3}$$

For $\alpha \neq 1$, one can note the operator has memory effect due to the kernel function $1/(t-\tau)^\alpha$.

In this paper, we consider the following fractional diffusion equation:

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$$ {}_a^C\mathrm{D}_t^\alpha u - \lambda\, {}_a^C\mathrm{D}_t^\beta u = D\frac{\partial^2 u}{\partial x^2} + g(u), \qquad 0<\beta<\alpha<1, \qquad \alpha = 2v, \qquad \beta = v \tag{4} $$

subjected to the condition:

$$ u(x,0) = f(x) = x, \qquad -\infty < x < +\infty \tag{5} $$

and λ is a constant.

Preliminaries

In the past ten years, many fruitful results [2-9] have been developed for the fractional diffusion equation. The famous Adomian decomposition method (ADM) is proposed by Adomian [10] for solving non-linear differential equations. Then it was successfully extended to fractional differential equations [11, 12]. However, the computation becomes more complicated and costs much time. Duan [13] and Duan *et al.* [14, 15] proposed an alternative way to calculate the Adomian polynomials and partly solved the difficulty arising in the solutions. Its extended version was reported in [16-18].

Definition 2.1. The fractional integral is defined by:

$$ {}_0I_t^\alpha u = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau, \qquad 0<\alpha, \qquad 0<t \tag{6} $$

Properties 2.2. Assume $N(u)$ as a non-linear term. Let $u_n = \Sigma_{k=0}^n \phi_k$ and apply the Adomian polynomials to treat or expand it as $\Sigma_{k=0}^\infty A_k$:

$$ A_k = \frac{1}{k}\sum_{j=0}^{k-1}(j+1)\phi_{j+1}\frac{\mathrm{d}A_{k-1-j}}{\mathrm{d}\phi_0} \tag{7} $$

Properties 2.3. The Leibniz integral law [1] holds:

$$ {}_0I_t^\alpha\, {}_0^C D_t^\alpha u(t) = u(t) - u(0), \qquad 0<t, \qquad 0<\alpha<1 \tag{8} $$

and

$$ {}_0I_t^\alpha\, {}_0^C\mathrm{D}_t^\beta u(t) = {}_0I_t^{\alpha-\beta}[u(t)-u(0)], \qquad 0<t, \qquad 0<\alpha<1 \tag{9} $$

In this way, one can obtain the integral equation and then the ADM or its modifications can be implemented.

Algorithms, results, and error analysis

Algorithms based on the Adomian polynomials

In this section, we investigate the reaction diffusion equation fractional order. Let's give the following steps.

(a) Take the integral (6) on both sides so that the integral equation with respect to fractional orders can be obtained:

$$ u(x,t) = u(x,0) + \lambda_0 I_t^{\alpha-\beta}[u(x,t)-u(x,0)] + {}_0I_t^\alpha\left[D\frac{\partial^2 u}{\partial x^2} + g(u)\right], \qquad 0<\beta<\alpha<1 \tag{10} $$

(b) According to [13-15], assume $u_n = \sum_{k=0}^{n} \phi_k(x) t^{kv}$ and $\phi_0(x) = f(x)$. It follows that:

$$\begin{cases} \phi_0 = f(x), \\ \phi_1 = 0, \\ \phi_k = \lambda \phi_{k-1} \dfrac{\Gamma[(k-1)v+1]}{\Gamma(kv+1)} + D\phi_{k-2,x,x} \dfrac{\Gamma[(k-2)v+1]}{\Gamma(kv+1)} + A_{k-2} \dfrac{\Gamma[(k-2)v+1]}{\Gamma(kv+1)}, \qquad k \geq 2 \end{cases} \tag{11}$$

Approximate solutions and error analysis

Consider the initial value $\phi_0(x) = f(x)$, the non-linear term $g(u) = Cu^2(1-u)$ and select the parameters $C = D = 0.2$ and $v = 0.4$. It follows that with the previous steps, we can have the approximate solutions. We only give the first two:

$$u_1 = x;$$
$$u_2 = x + 0.214734x^2(1-x)t^{0.8}$$
$$\ldots$$

We plot the eighth-order approximation in fig. 1.

Let's define the error function:

$$f = \ln \left| {}_a^C\mathrm{D}_t^\alpha u - \lambda\, {}_a^C\mathrm{D}_t^\beta u - D \frac{\partial^2 u}{\partial x^2} - g(u) \right| \tag{12}$$

Through the error analysis in fig. 2, we can see the result and the method are reliable which can be used in the fractional modeling.

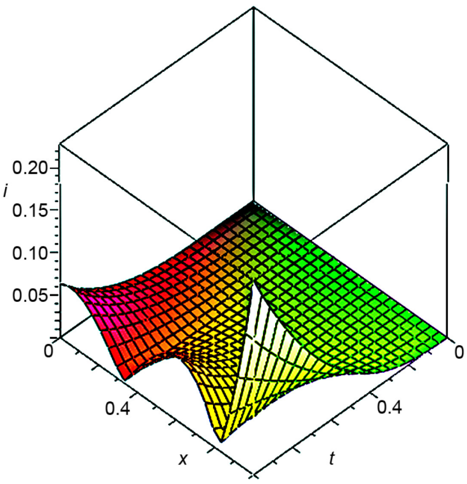

Figure 1. Non-linear diffusion behaviors for v = 0.4

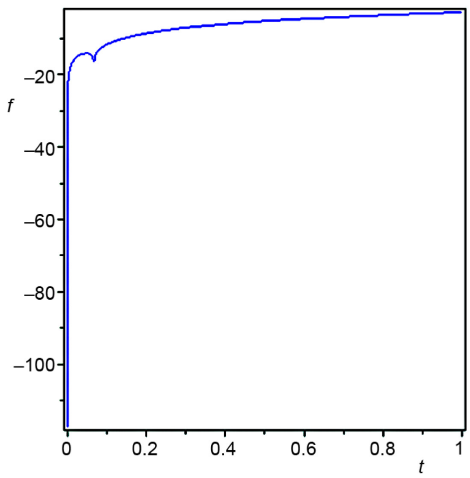

Figure 2. Error curve for v = 0.4 and x = 0.5

Take look at the other cases in figs. 3-5 where we vary the fractional orders. We can conclude that the larger the fractional order, the greater the concentration. The previous results are performed on the MAPLE.

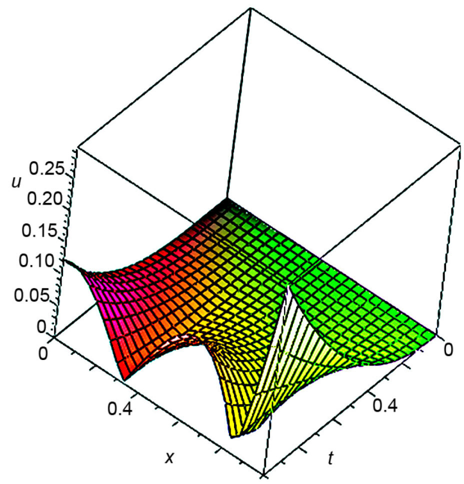

Figure 3. Non-linear diffusion behaviors for $v = 0.35$

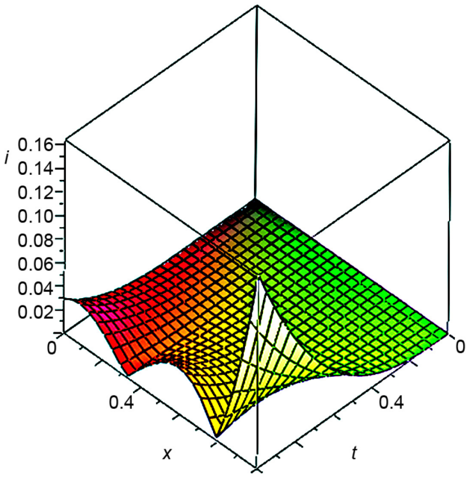

Figure 4. Non-linear diffusion behaviors for $v = 0.45$

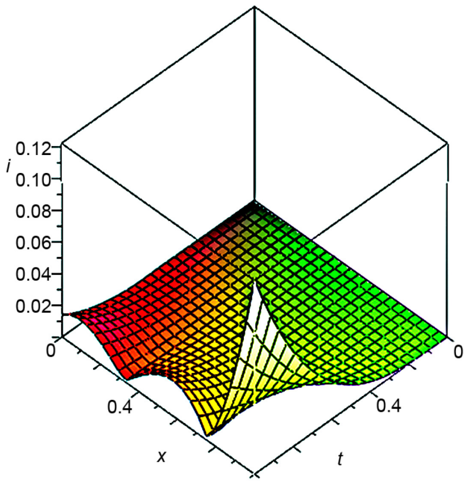

Figure 5. Non-linear diffusion behaviors for $v = 0.49$

Conclusions

In this paper, we analytically solve the fractional diffusion equation of two orders. The recurrence formula is derived based on the Taylor series and the Adomian polynomials. Then the approximate solutions are given by symbolic computation. We conclude the relationship between the diffusion concentration and the fractional orders. In our future work, we will discuss other initial boundary problems.

Nomenclature

D – diffusion coefficient, [m^2s^{-1}]
n – integer, [–]
t – time, [s]
u – concentration, [$molcm^{-3}$]
x – displacement, [cm]

Greek symbols

α – fractional order, [–]
β – fractional difference order, [–]
v – fractional order, [–]

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