# THREE-DIMENSIONAL AND TWO-PHASE NANOFLUID FLOW AND HEAT TRANSFER ANALYSIS OVER A STRETCHING INFINITE SOLAR PLATE

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Original scientific paper https://doi.org/10.2298/TSCI160614266H

In this work, 3-D and two-phase nanofluid flow and heat transfer is modeled over a stretching infinite solar plate. The governing equations are presented based on previous studies. The infinite boundary condition and shortcoming of traditional analytical collocation method have been overcome in our study by changing the problem into a finite boundary problem with a new analytical method called optimal collocation method. The accuracy of results is examined by fourth order Runge-Kutta numerical method. Effect of some parameters, Prandtl number, Schmidt number, Brownian motion parameter, thermophoresis parameter,  $\lambda=b/a$  (ratio of the stretching rate along y- to x-directions), and power-law index on the velocities, temperature, and nanoparticles concentration functions are discussed. As an important outcome of our 3-D model analysis, it is found that increase in thermophoretic forces can enhance the thickness of both thermal and nanoparticle volume fraction boundary-layers.

Key words: solar plate, nanofluid, nanoparticle concentration, infinite boundary, optimal collocation method

### Introduction

Nanofluids have been widely investigated to increase the thermal efficiency of the solar devices, such as for cavity [1], solar plates [2], and heat pipes [3]. The flow over moving or stationary solid surfaces has been a prime interest of researchers due to their various engineering applications when employing nanofluids. Except of experimental and numerical works in this field, some researchers worked analytically on the nanofluids treatment in solar applications. Khan *et al.* [4] analyzed the 3-D flow of nanofluid over an elastic sheet stretched non-linearly in two lateral directions under the solar radiation using Runge-Kutta method. In another work, Cregan *et al.* [5] considered both a radiative transport equation describing the propagation of solar radiation through the nanofluid and an energy equation for the steady-state, 2-D model of an inclined nanofluid-based direct absorption solar collector. Turkyilmazoglu [6] evaluated the

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alumina nanoparticles effect on thermal performance of a water based solar collector analytically. We have studied the transient vertical motion of a soluble particle in a Newtonian fluid media [7], motion of a spherical particle on a rotating parabola [8] or in a fluid forced vortex [9] by efficient mathematic methods. Additionally, Dogonchi *et al.* [10] investigated the motion of spherical solid particle in plane Couette Newtonian fluid flow.

Comparison of the single and two-phase modelling for the nanofluids has been considered by the researchers. For instance, Fard *et al.* [11] compared the results of the single phase and two-phase numerical methods for nanofluids in a circular tube. They reported that for Cu-water the average relative error between experimental data and CFD results based on single-phase model was 16% while for two-phase model it was only 8%. In another numerical study, Goktepe *et al.* [12] compared these two models for nanofluid convection at the entrance of a uniformly heated tube and found the same results. Mohyud-Din *et al.* [13] considered the 3-D heat and mass transfer with magnetic effects for the flow of a nanofluid between two parallel plates in a rotating system. They found that thermophoresis and Brownian motion parameters are directly related to heat transfer but are inversely related to concentration profile. The 3-D flow of nanofluids under the radiation (due to solar or *etc.*) has been analyzed by Hayat *et al.* [14]. Other related work can be found in literatures [15-22].

There are some simple and accurate analytical techniques for solving non-linear differential equations such as weighted residuals methods (WRM) and differential transformation method. Collocation method (CM), Galerkin method (GM), and least square method (LSM) are examples of the WRM which are introduced by Ozisik [23] for solving the heat transfer problems. Stern and Rasmussen [24] used CM for solving a third order linear differential equation. Vaferi *et al.* [25] have studied the feasibility of applying of an orthogonal CM to solve diffusivity equation in the radial transient flow system. Recently we have used LSM for heat transfer study through porous fins [26]. This accurate method has been applied to fully wet circular porous fin [27], semi-spherical porous fins [28], and straight solid and porous fins [29], *etc.* Gao and Duan [30] developed and analyzed least-squares approximations for the incompressible magneto-hydrodynamic equations. Ghasemi *et al.* [31] found that LSM is more appropriate than other analytical methods for solving the non-linear heat transfer equations.

It has been reviewed that nanofluids flow problem can be treated both by numerical [32] or optimization [33] methods. In the present study, an application of nanofluids for a solar plate is introduced. The 3-D and two-phase nanofluid flow and heat transfer analysis over a stretching infinite solar plate was studied. We aim to investigate the heat transfer and fluid-flow by an efficient technique for solving the non-linear governing equation. Considering the infinity boundary condition of the studied problem, an optimal collocation method (OCM) is used and some valuable results have been obtained.



Figure 1. Schematic of the nanofluid flow over a stretching solar plate

## **Problem description**

A nanofluid flow over a solar plate proposed as by Khan *et al.* [4] is shown in fig. 1. The flow is incompressible and induced due to plate stretched in two directions by non-linear functions. The plate is maintained at constant temperature and the mass flux of the nanoparticles at the wall is assumed to be zero. The 3-D governing equations will be [4]: Hatami, M., *et al.*: Three-Dimensional and Two-Phase Nanofluid Flow and Heat Transfer ... THERMAL SCIENCE: Year 2018, Vol. 22, No. 2, pp. 871-884

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = v_f \frac{\partial^2 u}{\partial z^2}$$
(2)

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = v_f \frac{\partial^2 v}{\partial z^2}$$
(3)

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + \tau \left[ D_B \frac{\partial C}{\partial z} \frac{\partial T}{\partial z} + \frac{D_T}{T_{\infty}} \left( \frac{\partial T}{\partial z} \right)^2 \right]$$
(4)

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_B \frac{\partial^2 C}{\partial z^2} + \left(\frac{D_T}{T_{\infty}}\right)\frac{\partial^2 T}{\partial z^2}$$
(5)

Here u and v are the velocities in the x- and y-directions, respectively, T – the temperature, C – the concentration,  $D_B$  – the Brownian diffusion coefficient of the diffusing species, and  $D_T$  – the thermophoretic diffusion coefficient. Because the plate is infinite and is stretched in two directions as described by non-linear functions, the relevant boundary conditions are:

$$u = u_w = a(x + y)^n, \quad v = v_w = b(x + y)^n$$
  

$$w = 0, \quad T = T_w, \quad D_B \frac{\partial C}{\partial z} + \frac{D_T}{T_\infty} \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = 0$$
  

$$u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad z \to 0$$
(6)

By introducing these parameters:

$$u = a(x+y)^{n} f'(\eta), \quad v = a(x+y)^{n} g'(\eta)$$

$$w = -\sqrt{av_{f}} (x+y)^{\frac{n-1}{2}} \left[ \frac{n+1}{2} (f+g) + \frac{n-1}{2} \eta (f'+g') \right]$$

$$\theta(\eta) = \frac{T-T_{\infty}}{T_{w}-T_{\infty}}, \quad \phi(\eta) = \frac{C-C_{\infty}}{C_{\infty}}, \quad \eta = \sqrt{\frac{a}{v_{f}}} (x+y)^{\frac{n-1}{2}} z$$
(7)

and substituting the previous variables into eqs. (1)-(5), one can get:

$$f''' + \frac{n+1}{2}(f+g)f'' - n(f'+g')f' = 0$$
(8)

$$g''' + \frac{n+1}{2}(f+g)g'' - n(f'+g')g' = 0$$
<sup>(9)</sup>

$$\frac{1}{\Pr}\theta'' + \frac{n+1}{2}(f+g)\theta' + Nb\phi'\theta' + Nt\theta'^2 = 0$$
(10)

$$\phi'' + \frac{n+1}{2} \operatorname{Sc}(f+g)\phi' + \frac{Nt}{Nb}\theta'' = 0$$
(11)

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These systems of non-linear equations should be solved by a powerful numerical or analytical method. In this study OCM is applied with these boundary conditions:

$$f(0) = 0, \ f'(0) = 1, \ g(0) = 0, \ g'(0) = \lambda, \ \theta(0) = 1, \ Nb\phi'(0) + Nt\theta'(0) = 0$$
  
$$f'(\infty) \to 0, \ g'(\infty) \to 0, \ \theta(\infty) \to 0, \ \phi(\infty) \to 0$$
(12)

where Pr is the Prandtl number, Sc – the Schmidt number, Nb – the Brownian motion parameter, Nt – the thermophoresis parameter,  $\lambda = b/a$  – the ratio of the stretching rate along y- to x-directions are parameters as the fined in [4].

## Collocation and optimal collocation method

The CM method is one of the approximation techniques for solving differential equations called the WRM. As the main idea of this method, a differential operator, D, is supposed to be acted on a function u to produce a function p [15]:

$$D[u(x)] = p(x) \tag{13}$$

We wish to approximate u by a function  $\tilde{u}$ , which is a linear combination of basic functions chosen from a linearly independent set. That is:

$$u = \tilde{u} = \sum_{i=1}^{n} c_i \varphi_i \tag{14}$$

Now, when substituted into the differential operator, D, the result of the operations is not p(x). Hence an error or residual will exist:

$$E(x) = R(x) = D[\tilde{u}(x) - p(x)] \neq 0$$
(15)

The target in the collocation is to force the residual to zero in some average sense over the domain. That is:

$$\int R(x)W_{i}(x) dx = 0 , i = 1, 2, ..., n$$
(16)

where the number of weight functions,  $W_i$ , is exactly equal to the number of unknown constants  $c_i$  in  $\tilde{u}$ . The result is a set of n algebraic equations for the unknown constants  $c_i$ . For the CM, the weighting functions are taken from the family of Dirac  $\delta$  functions in the domain. That is,  $w_i(x) = \delta(x - x_i)$ . The Dirac  $\delta$  function has the property of:

$$\delta(x - x_i) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwhise} \end{cases}$$
(17)

Residual function in eq. (15) must be forced to be zero at specific points. Further, we will use the OCM to solve eqs. (8)-(11). This method can be considered as a modification of the CM. To implement the method, the physical region  $\eta = [0, \infty)$  is transformed into the region  $\eta_1 = [0, \eta_{\infty 1}]$  and  $\eta_2 = [0, \eta_{\infty 2}]$  along the x- and y-directions for the hydrodynamic boundary-layer,  $\eta_3 = [0, \eta_{\infty 3}]$  and  $\eta_4 = [0, \eta_{\infty 4}]$  for the thermal and nanoparticle volume fraction boundary-layer, respectively. The  $\eta_{\infty 1}, \eta_{\infty 2}, \eta_{\infty 3}$ , and  $\eta_{\infty 4}$  are supposed to be sufficiently large and are the maximum values of  $\eta$  at the edge of the boundary-layer. The values of  $\eta_{\infty}$  change when we have variable parameters. Note that,  $\eta_{\infty}$  is a function of Sc, Pr, Nb, Nt,  $\lambda$ , and n determined as a part of solution. On the other hand, by introducing the following change of variable  $z_1=\eta_1/\eta_{\infty 1}, z_2=\eta_2/\eta_{\infty 2}, z_3=\eta_3/\eta_{\infty 3}$  and  $z_4=\eta_4/\eta_{\infty 4}$ , the problem transforms into the interval [0,1]

instead of  $[0, \eta_{\infty}]$ . Applying the previous transformations, the governing equations eqs. (8)-(11) can be transformed into the following form:

$$\frac{\frac{d^{3}}{dz_{1}^{3}}H(z_{1})}{\eta_{\omega_{1}}^{2}} + \frac{(n+1)\left[\eta_{\omega_{1}}H(z_{1}) + \eta_{\omega_{2}}K(z_{2})\right]\left[\frac{d^{2}}{dz_{1}^{2}}H(z_{1})\right]}{2\eta_{\omega_{1}}} - n\left[\frac{d}{dz_{1}}H(z_{1}) + \frac{d}{dz_{2}}K(z_{2})\right]\left[\frac{d}{dz_{1}}H(z_{1})\right] = 0$$
(18)

$$\frac{\frac{d^{3}}{dz_{2}^{3}}K(z_{2})}{\eta_{\infty 2}^{2}} + \frac{(n+1)\left[\eta_{\infty 1}H(z_{1}) + \eta_{\infty 2}K(z_{2})\right]\left[\frac{d^{2}}{dz_{2}^{2}}K(z_{2})\right]}{2\eta_{\infty 2}} - n\left[\frac{d}{dz_{1}}H(z_{1}) + \frac{d}{dz_{2}}K(z_{2})\right]\left[\frac{d}{dz_{2}}K(z_{2})\right] = 0$$
(19)

$$\frac{\frac{\mathrm{d}^{2}}{\mathrm{d}z_{3}^{2}}M(z_{3})}{\eta_{\infty 3}\operatorname{Pr}} + \left(\frac{n+1}{2}\right)\left[\eta_{\infty 1}H(z_{1}) + \eta_{\infty 2}K(z_{2})\right]\left[\frac{\mathrm{d}}{\mathrm{d}z_{3}}M(z_{3})\right] + Nb\left[\frac{\mathrm{d}}{\mathrm{d}z_{4}}N(z_{4})\right]\left[\frac{\mathrm{d}}{\mathrm{d}z_{3}}M(z_{3})\right] + Nt\left[\frac{\mathrm{d}}{\mathrm{d}z_{3}}M(z_{3})\right]^{2} = 0$$
(20)

$$\frac{\frac{\mathrm{d}^2}{\mathrm{d}z_4^2}N(z_4)}{\eta_{\infty 4}} + \left(\frac{n+1}{2}\right)\mathrm{Sc}\left[\eta_{\infty 1}H(z_1) + \eta_{\infty 2}K(z_2)\right]\left[\frac{\mathrm{d}}{\mathrm{d}z_4}N(z_4)\right] + \frac{Nt\left[\frac{\mathrm{d}^2}{\mathrm{d}z_3^2}M(z_3)\right]}{Nb\eta_{\infty 3}} = 0$$

$$(21)$$

where  $h(z_1) = f(\eta_1)/\eta_{\infty}$ ,  $k(z_2) = g(\eta_2)/\eta_{\infty}$ ,  $M(z_3) = \theta(\eta_3)/\eta_{\infty}$ ,  $N(z_4) = \phi(\eta_4)/\eta_{\infty}$  and the *prime* denotes the derivatives with respect to  $z \in [0,1]$ . Also, the boundary conditions can be transformed into:

$$z_1 = z_2 = z_3 = z_4 = 0 \implies H = 0, \ K = 0, \ H' = 1, \ K' = \lambda, \ M = \frac{1}{\eta_{\infty 3}}, \ NbM' + NtN' = 0 \ (22)$$

$$z_1 = z_2 = z_3 = z_4 = 1 \implies h' = 0, \ k' = 0, \ M = 0, \ N = 0$$
 (23)

In this method, the last boundary conditions are obtained by using the asymptotic conditions:

$$\eta \to \infty \implies f'' = 0, \ g'' = 0, \ \theta' = 0, \ \phi' = 0 \tag{24}$$

The extra boundary conditions (24) can be replaced by the conditions:

$$z_1 = z_2 = z_3 = z_4 = 1 \implies H'' = 0, \ K'' = 0, \ M' = 0, \ N' = 0$$
(25)

The asymptotic condition (25) is to be imposed for computing  $\eta_{\infty 1}$ ,  $\eta_{\infty 2}$ ,  $\eta_{\infty 3}$ , and  $\eta_{\infty 4}$ . We wish to obtain an approximate solution for this problem in the interval  $0 \le z \le 1$ . To construct a trial solution, we choose the basic function to polynomial in z. The trial solution contains undetermined coefficients c:

$$H(z_1) \coloneqq z_1 + c_1 z_1^2 + c_2 z_1^3 + c_3 z_1^4 + \dots + c_6 z_1^7$$
(26)

$$K(z_2) = \lambda z_2 + c_7 z_2^2 + c_8 z_2^3 + c_9 z_2^4 + \dots + c_{12} z_2^7$$
(27)

$$M(z_3) = \frac{1}{\eta_{\infty 3}} + c_{13}z_3 + c_{14}z_3^2 + c_{15}z_3^3 + \dots + c_{18}z_3^6$$
(28)

$$N(z_4) = c_{19} - \frac{Nt}{Nb}c_{13}z_4 + c_{21}z_4^2 + c_{22}z_4^3 + \dots + c_{25}z_4^6$$
(29)

The accuracy of the solution can be improved by increasing the number of its terms. Whereas the trial solution must satisfy the boundary conditions of (24) and (25) for all values of *c*, thus we have the following:

$$1 + 2c_1 + 3c_2 + \ldots + 7c_6 = 0 \tag{30}$$

$$\lambda + 2c_7 + 3c_8 + \ldots + 7c_{12} = 1 \tag{31}$$

$$\frac{1}{\eta_{\infty 3}} + c_{13} + c_{14} + \ldots + c_{18} = 0$$
(32)

$$c_{19} - \frac{Nt}{Nb}c_{13} + c_{21}\dots + c_{25} = 1$$
(33)

The  $\eta_{\infty_1}$ ,  $\eta_{\infty_3}$ ,  $\eta_{\infty_4}$ , and  $\eta_{\infty_4}$  in eqs. (18)-(21) can be calculated by using the extra boundary conditions given in eq. (25). This yields:

$$2c_1 + 6c_2 + 12c_3 \dots + 42c_6 = 0 \tag{34}$$

$$2c_7 + 6c_8 + 12c_9 + \ldots + 42c_{12} = 0 \tag{35}$$

$$c_{13} + 2c_{14} + 3c_{15} + \ldots + 6c_{18} = 0$$
(36)

$$-\frac{Nt}{Nb}c_{13} + 2c_{21} + \dots + 6c_{25} = 0$$
(37)

By introducing  $g(z_1)$ ,  $h(z_2)$ ,  $M(z_3)$ , and  $N(z_4)$  to differential equations (18)-(21), residual functions will be found:

$$R_{1}(c_{1},c_{2},...,c_{12},\eta_{\infty 1},\eta_{\infty 2},z_{1},z_{2}), \quad R_{2}(c_{1},c_{2},...,c_{12},\eta_{\infty 1},\eta_{\infty 2},z_{1},z_{2}),$$
  
$$R_{3}(c_{1},c_{2},...,c_{25},\eta_{\infty 1},\eta_{\infty 2},\eta_{\infty 3},\eta_{\infty 4},z_{1},z_{2},z_{3},z_{4})W_{i}$$

and

$$R_{4}(c_{1},c_{2},...,c_{25},\eta_{\infty 1},\eta_{\infty 2},\eta_{\infty 3},\eta_{\infty 4},z_{1},z_{2},z_{3},z_{4})$$

In the OCM, the numbers of weight functions,  $W_i$ , are:

$$n_{w_i} = n_{c_i} - n_b \tag{38}$$

where  $n_{c_i}$  are the numbers of unknown constants and  $c_i$ , and  $n_b$  are the number of equations that satisfy the boundary conditions. On the other hand, the residual function must be close to zero. For reaching to this aim, specific points in the domain  $z \in [0,1]$  should be chosen. These points are:

$$R_1\left(\frac{1}{6}\right) = 0, \qquad R_1\left(\frac{2}{6}\right) = 0, \qquad \dots, \qquad R_1\left(\frac{5}{6}\right) = 0$$
 (39)

$$R_2\left(\frac{1}{6}\right) = 0, \qquad R_2\left(\frac{2}{6}\right) = 0, \qquad \dots, \qquad R_2\left(\frac{5}{6}\right) = 0$$
 (40)

$$R_3\left(\frac{1}{6}\right) = 0, \qquad R_3\left(\frac{2}{6}\right) = 0, \qquad \dots, \qquad R_3\left(\frac{5}{6}\right) = 0$$
 (41)

$$R_4\left(\frac{1}{6}\right) = 0, \qquad R_4\left(\frac{2}{6}\right) = 0, \qquad \dots, \qquad R_4\left(\frac{5}{6}\right) = 0$$
 (42)

In this problem, we have a set of nine algebraic equations: four equations, eqs. (30)-(33), which satisfy the boundary conditions, four equations, eqs. (34)-(37), which satisfy the extra boundary conditions and twenty equations, eqs. (39)-(42), with residual function kept close to zero. By solving this system of equations, unknown coefficients  $c_i$ ,  $\eta_{\infty 1}$ ,  $\eta_{\infty 3}$ ,  $\eta_{\infty 4}$ , and  $\eta_{\infty 4}$  will be determined. Finally after specifying these unknown parameters, the velocity and temperature distribution will be determined. For example, using OCM for a nanofluid with Sc = 1, Pr = 25, Nb = 0.1, Nt = 0.1,  $\lambda = 0.5$ , and  $n = 3 f(\eta)$ ,  $g(\eta)$ ,  $\theta(\eta)$ , and  $\phi(\eta)$  are as shown in eq. (43). In a similar manner, we will obtain other solutions for cases of different parameters. The results will be presented in next section.

$$f(\eta) = 1.000000000\eta - 0.9682222966\eta^{2} + 0.6478596724\eta^{3} + \dots + 0.0018627118\eta^{7}$$

$$g(\eta) = 0.500000003\eta - 0.4841111523\eta^{2} + 0.3239298503\eta^{3} + \dots + 0.0009313565\eta^{7}$$

$$\theta(\eta) = 1.0000000000 - 19.3130049200\eta + 154.9379597000\eta^{2} + \dots + 987.7535926000\eta^{6}$$

$$\phi(\eta) = -0.9615385160 + 19.3130049300\eta - 161.1354819000\eta^{2} + \dots - 1201.7536980000\eta^{6}$$
(43)

## **Results and discussion**

In order to ensure the accuracy of the present results, at first we compare our results corresponding to the shear stress at the surface, *i. e.*, f''(0) and g''(0), for various values of n and  $\lambda$  with the available published results of Khan *et al.* [4] and presented in tab. 1. This table confirms that an excellent agreement exists between the presented analytical method and previous shooting method. For better perception, a sample data and absolute errors is presented for other parameter values and velocity profiles in x and y directions via tab. 2. This table also confirms the high accuracy of our described method. Here, the influences of mathematical parameters appeared in the mathematical section on velocities, temperature and nanoparticle volume fraction profiles are discussed physically. Figure 2 demonstrate the effect of the power-law index n, both the dimensionless velocity in the x- and y-directions decreases correspondingly.

Also, it can be seen that the decrement of velocity in the x- and y-direction are equal roughly. For example, both the velocities in the x- and y-direction are decreased by about 60%

Table 1. Comparison between OCM and numerical shooting [4] in predicting f''(0) and g''(0) values

n	λ	-f''(0) shooting [4]	OCM	OCM	-g''(0) shooting [4]	OCM	Error
1	0	1	0.992650	0.007350	0.000000	0.000000	0.000000
1	0.5	1.224745	1.215743	0.009002	0.612372	0.607871	0.004501
1	1	1.414214	1.403819	0.010395	1.414214	1.403819	0.010395
3	0	1.624356	1.606584	0.017772	0.000000	0.000000	0.000000
3	0.5	1.989422	1.936444	0.052978	0.994711	0.968222	0.026489
3	1	2.297186	2.236013	0.061173	2.297186	2.236013	0.061173

Table 2. Comparison between OCM and numerical results when  $\lambda = 0.5$ , n = 3

η	Numerical	$f(\eta)$ OCM	Error	Numerical	$g'(\eta)$ OCM	Error
0	1.000000	1.000000	0.000000	0.500000	0.500000	0.000000
0.1	0.821284	0.824577	0.003293	0.410642	0.412288	0.001646
0.2	0.677489	0.681129	0.003640	0.338745	0.340565	0.001820
0.3	0.560912	0.563781	0.002869	0.280456	0.281890	0.001434
0.4	0.465764	0.467639	0.001875	0.232882	0.233819	0.000937
0.5	0.387641	0.388668	0.001026	0.193821	0.194334	0.000513
0.6	0.323159	0.323571	0.000412	0.161579	0.161786	0.000206
0.7	0.269685	0.269681	0.000004	0.134842	0.134840	0.000002
0.8	0.225157	0.224858	0.000299	0.112578	0.112429	0.000149
0.9	0.187941	0.187400	0.000541	0.093970	0.093700	0.000271
1	0.156735	0.155962	0.000774	0.078368	0.077981	0.000387
1.1	0.130494	0.129483	0.001011	0.065247	0.064741	0.000506
1.2	0.108371	0.107124	0.001247	0.054185	0.053562	0.000623
1.3	0.089677	0.088215	0.001462	0.044839	0.044107	0.000731
1.4	0.073850	0.072210	0.001640	0.036925	0.036105	0.000820
1.5	0.060426	0.058654	0.001771	0.030213	0.029327	0.000886
1.6	0.049022	0.047157	0.001865	0.024511	0.023579	0.000932
1.7	0.039321	0.037379	0.001942	0.019661	0.018690	0.000971
1.8	0.031059	0.029023	0.002037	0.015530	0.014511	0.001018
1.9	0.024016	0.021837	0.002178	0.012008	0.010919	0.001089
2	0.018005	0.015632	0.002373	0.009003	0.007816	0.001187
2.1	0.012873	0.010296	0.002577	0.006436	0.005148	0.001288
2.2	0.008487	0.005830	0.002657	0.004244	0.002915	0.001329
2.3	0.004738	0.002390	0.002349	0.002369	0.001195	0.001174
2.4	0.001533	0.000331	0.001201	0.000766	0.000166	0.000601

at  $\eta = 1$  when the n change from 1 to 4. Our results show that both velocity profiles are decreasing function of power index. The effect of this power index on temperature and nanoparticles volume fraction boundary-layer profile is depicted via fig. 3. It is clear that thermal boundary-layer will be thinner for larger *n* values while nanoparticles concentration profile become thicker, both of which augments, in turn, the rate of heat transfer from the sheet. It can also be found that the greater the n value is, the faster the decline of  $\theta$  will be. It is proposed that the increasing *n* can enhance the convective properties of the fluid since it will increase the deformation by the shear stress from the wall to the fluid.



Figure 2. Influence of power-law index, *n*, on *f*' and *g*' functions (for color image see journal web site)



**Figure 3. Influence of power-law index**, *n*, **on temperature and nanoparticles volume fraction functions** (*for color image see journal web site*)

The influence of stretching rate ratio,  $\lambda$ , on the velocities components and temperature/nanoparticles concentration is presented in figs. 4 and 5, respectively. Physically, increase in stretching rate ratio, or the large values of  $\lambda$  (= b/a), means either increase in b or decrease in a, so it is expected to further lead to acceleration of the downward flow along the vertical direction, and as a result, a decrease in x-component and increase in y-component velocity profiles would occur. Moreover, increasing the stretching rate ratio makes a colder fluid-flow due to higher heat transfer process as can be found in fig. 5. Consequently, the thermal boundary-layer becomes thinner and both temperature and concentration profiles decrease.

As is well known, heat transfer rate is a decreasing function of *Nt* and Schmidt number. A higher Prandtl number fluid possesses stronger convection as compared to pure conduction and effective in transferring energy through unit area. The reduced Nusselt number therefore increases with an increase in Prandtl number. Figure 6 represents the effect of Prandtl



Figure 4. Influence of stretching rate ration,  $\lambda$ , on f' and g' functions (for color image see journal web site)



Figure 5. Influence of stretching rate ration,  $\lambda$ , on temperature and nanoparticles volume fraction functions (for color image see journal web site)

number on the thermal boundary-layer. As shown in this figure, temperature,  $\theta$ , decreases and becomes very close to the ambient temperature and the slope of temperature distribution near wall become steeper with an increment of Prandtl number. As Khan *et al.* [4] reported, an increase in Prandtl number accompanies with weaker thermal diffusivity and restricts the heat from flowing deeper into the nanofluid, so thermal boundary-layer becomes decreased with an augmentation of Prandtl number.

Schmidt number was defined as the ratio of momentum diffusivity to mass diffusivity, so an increase in Schmidt number also corresponds to the decrease of the Brownian diffusion coefficient,  $D_B$ . In accordance with Kuznetsov and Nield [34], the effect of increase in Brownian motion parameter on the temperature profile for natural convective boundary-layer flow of a nanofluid past a vertical plate is negligible. Similar conclusion could be made for our case. However, Brownian motion parameter could make a great difference for nanoparticle distribution in nanofluid. As shown in fig. 7, the smaller  $D_B$ , corresponding to higher Schmidt number,



image see journal web site)

nanoparticles volume fraction function (for color image see journal web site)

could result in more nanoparticle penetration due to higher momentum diffusivity which in turn gives rise to the shorter penetration depth of nanoparticle volume fraction,  $\phi$ . The thermophoretic force is the force that diffuses nanoparticles into ambient flow and due to the existence of the increased amount of the nanoparticles in the fluid the temperature gradient will be smaller. Thus, increase in Nt can lead to thicker thermal boundary-layer and higher nanoparticles volume fractions along z-direction, *i. e.*, higher  $\phi$ , as shown in fig. 8. Also, it can be seen that the temperature profiles become less steep with the increment of Nt. The observed curve characteristic is supposed to be due to the decrease in Nusselt number.

Finally, the effect of Brownian motion parameter, Nb, on nanoparticles volume fraction function is shown in fig. 9. This figure reveals that Brownian motion parameter has an inverse effect compared to Nt. Actually, an increase in Nb can lead to the decrease in nanoparticles concentration. Therefore,  $\phi$  will decrease upon the increasing of the Brownian motion param-



Figure 8. Effect of thermophoresis parameter, Nt, on temperature and nanoparticles volume fraction functions (for color image see journal web site)



Figure 9. Effect of Brownian motion parameter, Nb, on nanoparticles volume fraction function (for color image see journal web site)

eter *Nb*. The similar trend has been observed by Mustafa [35]. In their study, the laminar axisymmetric flow of nanofluid over a non-linearly stretching sheet has been studied. The model used for nanofluid considers the simultaneous effects of Brownian motion and thermophoretic diffusion of nanoparticles.

### Conclusion

In this study, a new analytical method called OCM have been successfully applied to find the solution of 3-D modelling of heat transfer for two-phase nanofluids flow over an infinite stretching solar plate. Due to infinite boundary condition and shortcoming of traditional analytical CM method, by a variation in the variables, the problem was changed to a finite boundary problem and solved by described method. The results show that OCM results are in excellent agreement with those of numerical solution. By a parametric study, it

was found that the velocity along the x- and y-directions decrease with the increase in power-law index n, so shear stresses at the wall are decreasing functions of power-law index. Additionally, the effect of some parameters, Prandtl number, Schmidt number, Brownian motion parameter, thermophoresis parameter, ratio of the stretching rate,  $\lambda = b/a$ , along y- to x-directions, and power-law index, n, appeared in the mathematical section on the velocities, temperature and nanoparticles concentration functions is discussed. As an important outcome of our 3-D model analysis, results show increase in stretching rate ratio, or the large values of  $\lambda (= b/a)$ , means either increase in b or decrease in a, it is thus expected to further lead to the acceleration of the downward flow along the vertical direction.

## Acknowledgment

The authors gratefully acknowledge the financial support of the National Natural Science Foundation of China (No. 51776165, 51502240) and the Natural Science Foundation of Jiangsu Province (No. BK20150378). This work was also supported by the China Fundamental Research Funds for the Central Universities.

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