REMARKS ON A GREEN FUNCTIONS APPROACH TO DIFFUSION MODELS WITH SINGULAR KERNELS IN FADING MEMORIES

by

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Diffusion problems with singulars kernels in fading memories are very interesting physical problem that have attracted attention of many researchers. In this paper, we aim to provide exact solutions of these problems using the green function method with some integral transform operators. The singular kernel used in this paper is based upon the power law function, which is used to construct the well-known Riemann-Liouville derivative with fractional order.

Key words: *diffusion models, singular kernel, green function, Laplace transform*

Introduction

The spread of molecules from an area of high-pitched of intensity to an area of little concentration has been a subject of many investigations within the past decade, due to the complexities of this movement within various types of media [1-3]. Deprived of reservation, this theory of diffusion is extensively employed across many fields including but not limited to finance with the spread of mankind, also with the concept of price values: in sociology, biology, physic, and chemistry [4-6]. In all these fields, the main idea is spreading out from an area of prohibitive concentration of that object. It is important to mention that, there exists two different ways of initiating the concept of diffusion including, a phenomenological methodology which of course starts with Fick's law of spreading together with their mathematical results and an atomistic approach which considers the random walk of the diffusion particles. We shall recall that in the scope of time, the theory underpinning the diffusion in solids was long employed before that of diffusion itself. For instance, one can remember the spreading of colours of stained glass also the Chinese ceramics.

Also in the contemporary scholarships, many researchers have devoted their focus in enhancing models for different type of diffusion. We shall start by quoting the example of Thomas Graham investigation that consisted in diffusion of gases. Due to the wide applicability of this concept, many researchers have suggested new analytical methods to solve diffusions models, for instance: Hristov [7] applied the integral balance method to solve model with elastic effect and that with memory kernel including a viscous damping, and many other results can be found in the following investigations [9-12].

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In this section, we start the study of a simple parabolic mathematical equation able to portray the rigid heat conductors and relaxing viscoelasticity with fading memory. The mathematical equation under investigation here is given:

$$\frac{\partial u(x,t)}{\partial t} = {}_0^C\mathrm{D}_t^\eta \left\{ \frac{\partial^2 [u(x,t)]}{\partial x^2} \right\} a_2,$$
$${}_0^C\mathrm{D}_t^\eta [f(t)] = \frac{1}{\Gamma(1-\eta)} \int_0^t (t-y)^{-\eta} \frac{\mathrm{d}}{\mathrm{d}y} f(y) \mathrm{d}y \tag{1}$$

where η is the fractional order in [0, 1]. Here we assume the following initial-boundary conditions:

$$u(x,0) = f(x), \quad u(0,t) = g(t), \quad u_x(x,0) = u_{xx}(x,0) = 0 \tag{2}$$

The integral operator considered used in this study is given:

$$L[f(x)](s) = \int_0^\infty f(x) \exp(-sx)\, \mathrm{d}x \tag{3}$$

Therefore, applying the Laplace transform on both sides of eq. (1), we get:

$$su(x,s) - u(x,0) = s^\alpha \frac{\partial^2 u(x,s)}{\partial x^2} a_2 \tag{4}$$

The next operator that will be used here is given:

$$F[y(t)] = \int_{-\infty}^{\infty} y(t) \exp(-2\pi i w t)\, \mathrm{d}t \tag{5}$$

Then applying the Fourier transform on both sides of eq. (4), we get:

$$s\tilde{u}(w,s) - \tilde{u}(w,0) = -a_2 w^2 s^\alpha \tilde{u}(w,s)$$
$$\tilde{u}(w,s) = \frac{\tilde{u}(w,0)}{s - w^2 s^\alpha a_2} \tag{6}$$
$$\tilde{u}(w,s) = \frac{1}{s^\alpha} \frac{\tilde{u}(w,0)}{s^{1-\alpha} - w^2 a_2}$$

Thus, an application of the inverse Laplace transform on both sides of eq. (6) produces:

$$\tilde{u}(w,t) = L^{-1}[\tilde{u}(w,s)] = L^{-1}\left[\frac{1}{s^\alpha} \frac{\tilde{u}(w,0)}{s^{1-\alpha} - w^2 a_2} \right] \tag{7}$$

Then using the convolution theorem for Laplace transform, we obtain:

$$\tilde{u}(w,t) = L^{-1}[\tilde{u}(w,s)] = \frac{\tilde{u}(w,0)}{\Gamma(\alpha)} \int_0^t (t-j)^{\alpha-1} j^{\alpha-2} E_{\alpha-1,\alpha-1}\left(a_2 w^2 j^{\alpha-1}\right) \mathrm{d}j \tag{8}$$

The previous solution is the exact solution in Fourier space, now to obtain the solution in real space we apply the inverse Fourier transform in both sides of eq. (8) to obtain:

$$u(x,t)=F^{-1}\{L^{-1}[\tilde{u}(w,s)]\}=F^{-1}\left[\frac{\tilde{u}(w,0)}{\Gamma(\alpha)}\int_0^t (t-j)^{\alpha-1}j^{\alpha-2}E_{\alpha-1,\alpha-1}(a_2w^2j^{\alpha-1})\mathrm{d}j\right]x \quad (9)$$

However, the inverse can not be obtained straightforward. The generalized Mittag-Leffler can be reformulated in series:

$$E_{\alpha-1,\alpha-1}\left(a_2w^2j^{\alpha-1}\right)=\sum_{l=0}^{\infty}\frac{\left(a_2w^2j^{\alpha-1}\right)^{\alpha-1}}{\Gamma\left[l\left(\alpha-1\right)+\alpha-1\right]} \quad (10)$$

Replacing eq. (10) into eq. (9) yields:

$$\begin{aligned}u(x,t)&=F^{-1}\left\{\frac{\tilde{u}(w,0)}{\Gamma(\alpha)}\int_0^t\left(t-j\right)^{\alpha-1}j^{\alpha-2}\sum_{l=0}^{\infty}\frac{\left(a_2w^2j^{\alpha-1}\right)^{l}}{\Gamma\left[l\left(\alpha-1\right)+\alpha-1\right]}\mathrm{d}j\right\}x=\\&=F^{-1}\left\{\frac{\tilde{u}(w,0)}{\Gamma(\alpha)}\sum_{l=0}^{\infty}\frac{a_2^lw^{2l}}{\Gamma\left[l\left(\alpha-1\right)+\alpha-1\right]}\int_0^t\left(t-j\right)^{\alpha-1}j^{(\alpha-1)l+\alpha-2}\mathrm{d}j\right\}=\\&=F^{-1}\left\{\sum_{l=0}^{\infty}\frac{\tilde{u}(w,0)a_2^lw^{2l}}{\Gamma\left[l\left(\alpha-1\right)+2\alpha-1\right]}t^{2\alpha+(\alpha-1)l-2}\right\}=\sum_{l=0}^{\infty}\frac{a_2^l}{\Gamma\left[l\left(\alpha-1\right)+2\alpha-1\right]}t^{2\alpha+(\alpha-1)l-2}\\&\int_0^x f(x-v)\frac{\mathrm{i}^{1-2l}\left|x\right|^{-1-2l}\Gamma(1+2l)\{[(-i)^{2l}-i^{2l}]\cos(l)\,\mathrm{sgn}(x)+\mathrm{i}[(-i)^{2l}+i^{2l}]\sin(l)\}}{\sqrt{2}}\mathrm{d}x\end{aligned} \quad (11)$$

Equation (11) is the exact solution of the simple case of diffusion with elastic influence.

Diffusion models with memory kernel including a viscous damping.

In this section, we examine the more general equation where the relaxation can be shown in terms of fractional derivative. The transient flow of second grade viscoelastic non-solidified can be moulded by means of the leading mathematical formulation, which in a spreading form is articulated:

$$\frac{\partial u(x,t)}{\partial t}=v\frac{\partial^2u(x,t)}{\partial x^2}+{}_0^C\mathrm{D}_t^{\varphi}\left[\frac{\partial^2u(x,t)}{\partial x^2}\right]p,\quad 0<\varphi\leq 1 \quad (12)$$

and

$$u(x,0)=f(x),\quad u(0,t)=f(t),\quad u_{xx}(x,0)=u_{xx}(0,t)=0$$

Using the integral balance method, Hristov presented an approximate solution of the eq. (12) [7]. Many other scholars in this field have also examined the previous equation when the time fractional derivative is replaced by the local derivative, which is a very easy case to handle. However, there is exact solution of this equation in case of fractional order derivative. In this section, we shall use the Laplace transform together with Fourier transform to derive the exact solution of eq. (12).

Analytical solution via Laplace Fourier method

Applying the Laplace transform in time component in eq. (12), we obtain:

$$su(x,s)-u(x,0)=v\frac{\partial^2 u(x,s)}{\partial x^2}+s^{\varphi}\left[\frac{\partial^2 u(x,s)}{\partial x^2}\right]p \tag{13}$$

Thus applying the Fourier transform operator on both sides of eq. (13) yields:

$$\begin{gathered}s\tilde{u}(w,s)-\tilde{u}(w,0)=-vw^2\tilde{u}(w,s)-ps^{\varphi}w^2\tilde{u}(w,s)\\ \tilde{u}(w,s)\left(s+vw^2+ps^{\varphi}w^2\right)=\tilde{u}(w,0)\\ \tilde{u}(w,s)=\frac{\tilde{u}(w,0)}{\left(s+vw^2+ps^{\varphi}w^2\right)}=\frac{\tilde{u}(w,0)}{\dfrac{s}{v+ps^{\varphi}}+w^2}\frac{1}{v+ps^{\varphi}}\end{gathered} \tag{14}$$

Therefore, an application of inverse Fourier transforms on both sides of eq. (14) provides:

$$u(x,s)=\frac{1}{v+ps^{\varphi}}F^{-1}\left[\frac{\tilde{u}(w,0)}{\dfrac{s}{v+ps^{\varphi}}+w^2}\right] \tag{15}$$

Applying the convolution theorem for Fourier transform, we get:

$$u(x,s)=\frac{1}{v+ps^{\varphi}}\int_0^x f(x-y)\left\{\begin{matrix}\dfrac{1}{2\sqrt{a}}\mathrm{e}^{-\sqrt{a}y}\sqrt{\dfrac{1}{2}}\left[\left(-1+\mathrm{e}^{2\sqrt{a}y}\right)\mathrm{sgn}(t)\{-1+\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}\right]+\\ +2\left(\begin{matrix}\mathrm{e}^{2\sqrt{a}y}\mathit{HeavisideTheta}\{-y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}+\\ +\mathit{HeavisideTheta}\{y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\end{matrix}\right)\end{matrix}\right\}\mathrm{d}y$$

$$a=\frac{s}{v+ps^{\varphi}}$$

The exact solution is then obtained only by applying the inverse Laplace transform on both sides to obtain:

$$u(x,t)=$$

$$=L^{-1}\left\{\frac{1}{v+ps^{\varphi}}\int_0^x f(x-y)\left\{\begin{matrix}\dfrac{1}{2\sqrt{a}}\mathrm{e}^{-\sqrt{a}y}\sqrt{\dfrac{a}{2}}\left[\left(-1+\mathrm{e}^{2\sqrt{a}y}\right)\mathrm{sgn}(t)\{-1+\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}\right]+\\ +2\left(\begin{matrix}\mathrm{e}^{2\sqrt{a}y}\mathit{HeavisideTheta}\{-y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}+\\ +\mathit{HeavisideTheta}\{y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\end{matrix}\right)\end{matrix}\right\}\right\}\mathrm{d}y$$

For simplicity, we let:

$$h(x,t)=$$

$$=L^{-1}\left\{\int_0^x f(x-y)\left\{\begin{matrix}\dfrac{1}{2\sqrt{a}}\mathrm{e}^{-\sqrt{a}y}\sqrt{\dfrac{a}{2}}\left[\left(-1+\mathrm{e}^{2\sqrt{a}y}\right)\mathrm{sgn}(t)\{-1+\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}\right]+\\ +2\left(\begin{matrix}\mathrm{e}^{2\sqrt{a}y}\mathit{HeavisideTheta}\{-y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\}+\\ +\mathit{HeavisideTheta}\{y\,\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\})\mathrm{sgn}[\mathrm{Re}(\sqrt{a})]\end{matrix}\right)\end{matrix}\right\}\right\}\mathrm{d}y \tag{16}$$

However, the inverse Laplace transform of $1/(v+ps^{j})$ is given:

$$L^{-1}\left(\frac{1}{v+ps^{\varphi}}\right)=p^{-1}L^{-1}\left(\frac{1}{\dfrac{v}{p}+s^{\varphi}}\right)=p^{-1}t^{\varphi-1}E_{\varphi,\varphi}\left(-\frac{v}{p}t^{\varphi}\right) \tag{17}$$

By convolution theorem for Laplace transform, we obtain the exact solution of the equation under investigation:

$$u(x,t)=\int_0^t h(x,t-k)p^{-1}k^{\varphi-1}E_{\varphi,\varphi}\left\{-\frac{v}{p}k^{\varphi}\right\}\mathrm{d}k \tag{18}$$

Analytical solution via double-Laplace transform method

In this section, we employ the double-Laplace transform operator to obtain an equivalent solution of eq. (12). Thus, applying the double-Laplace transform on space and time component, we get:

$$\begin{gathered}
sU(l,s)-U(l,0)=vl^2U(j,s)+ps^jl^2uU(p,s)\\
U(p,s)\left\{s-vl^2-ps^jl^2\right\}=U(l,0)\\
U(l,s)=\frac{U(w,0)}{\left\{s-vl^2-ps^jl^2\right\}}=\frac{U(l,0)}{\dfrac{s}{v+ps^j}-l^2}\frac{1}{v+ps^j}=\\
=-\frac{U(l,0)}{l^2-\dfrac{s}{v+ps^j}}\frac{1}{v+ps^j}
\end{gathered} \tag{19}$$

Applying the inverse Laplace in space component we obtain:

$$u(x,s)=-\frac{1}{v+ps^{\varphi}}\int_0^x f(x-y)\frac{\sin\left(\sqrt{\dfrac{s}{v+ps^{\varphi}}}y\right)}{\sqrt{\dfrac{s}{v+ps^{\varphi}}}}\mathrm{d}y=$$

$$=\frac{1}{v+ps^{\varphi}}\int_0^x f(x-y)\sum_{i=0}^{\infty}\left\{\frac{\left(-\sqrt{\dfrac{s}{v+ps^{\varphi}}}\right)^{2i}}{(2i+1)!}\right\}y^{2i+1}\mathrm{d}y=$$

$$=\int_0^x f(x-y)\sum_{i=0}^{\infty}\left\{\frac{\dfrac{1}{v+ps^{\varphi}}\left(\dfrac{s}{v+ps^{\varphi}}\right)^{i}}{(2i+1)!}\right\}y^{2i+1}\mathrm{d}y=$$

$$=\int_0^x f(x-y)\sum_{i=0}^{\infty}\left[\frac{s^i}{(2i+1)!}\right]\left(\frac{1}{v+ps^{\varphi}}\right)^{i-1} y^{2i+1}\mathrm{d}y=$$

$$=\sum_{i=0}^{\infty}\left[\frac{s^i}{(2i+1)!}\right]\left(\frac{1}{v+ps^{\varphi}}\right)^{i-1}\int_0^x f(x-y)y^{2i+1}\mathrm{d}y=$$

$$=\sum_{i=0}^{\infty}\left[\frac{s^i}{(2i+1)!}\right]\left(\frac{p}{v}\right)^{i-1}\left(\frac{1}{\frac{v}{p}+s^{\varphi}}\right)^{i-1}\int_0^x f(x-y)y^{2i+1}\mathrm{d}y \tag{20}$$

But we have the following relation:

$$L^{-1}\left[\left(\frac{1}{\frac{v}{p}+s^{\varphi}}\right)^{i-1}\right]=\frac{1}{(i-2)!}t^{\varphi(i-1)+\varphi-1}E_{\varphi,\varphi}^{(i-1)}\left(-\frac{v}{p}t^{\varphi}\right) \tag{21}$$

also

$$L^{-1}\left\{s^i\right\}=\frac{t^{-1-i}}{\Gamma(-i)} \tag{22}$$

By convolution theorem we have:

$$u(x,t)=\sum_{i=0}^{\infty}\left[\frac{1^i}{(2i+1)!}\right]\left(\frac{p}{v}\right)^{i-1}\int_0^t\left\{\frac{(t-j)^{-1-i}}{\Gamma(-i)(i-1)!}j^{(i-1)\varphi+\varphi-1}E_{\varphi,\varphi}^{(i-1)}\left(-\frac{v}{p}j^{\varphi}\right)\right\}\mathrm{d}j\int_0^x f(x-y)y^{2i+1}\mathrm{d}y$$

The previous equation is the exact solution of eq. (12).

Conclusion

Partial differential equations constructed with the concept of fractional differentiation are suitable to model real world problems. However, these equations are sometime very difficult to handle analytically. For those physical problems that can be described with linear fractional partial differential equations, an attempt to find an exact solution is done using some technique like Laplace, Fourier, Sumudu, and Mellin transforms method. The use of these integral transforms commonly leads to obtaining the green function. In this paper using both Laplace and Fourier transform with the green function technique, we constructed the exact solutions of some diffusion problems with singular kernels.

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