TRAVELLING WAVE SOLUTIONS FOR A SURFACE WAVE EQUATION IN FLUID MECHANICS

by

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This paper considers a non-linear wave equation arising in fluid mechanics. The exact traveling wave solutions of this equation are given by using G'/G-expansion method. Tthis process can be reduced to solve a system of determining equations, which is large and difficult. To reduce this process, we used Wu elimination method. Example shows that this method is effective.

Key words: *traveling wave solutions, Wu elimination method, fluid mechanics*

Introduction

Recently, many powerful methods have been established and improved to seek exact solutions of non-linear evolution equations (NLEE) which describe non-linear phenomena arising in physics, mechanics, non-linear optic, and other fields. Some of these methods include auxiliary equation method [1], Clarkson-Kruskal direct method [1], modified variational iteration method [2], improved extended tg-function method [3], and so on.

In this paper, we will consider a surface wave equation [4]:

$$u_t \quad a_0 u_x \quad a_1 u u_x \quad a_2 u_{xxx} \quad b_0 u_{xx} \quad b_1 (u u_x)_x \quad b_2 u_{xxx} \quad 0 \tag{1}$$

which describes oscillatory Rayleigh-Marangoni instability in a liquid layer with free boundary.

The *G'*/*G*-expansion method has become widely used to search for various exact solutions of NLEE recently [5-7], the value of this method is that one treats non-linear problems by essentially linear methods. We first treat the governing eq. (1) by G'/G-expansion method, and obtain a large system of algebraic equations, which is difficult to solve, then we use Wu elimination method to solve this problem.

Application of the G'/G-expansion method for the eq. (1)

Let's assume the traveling wave solution of eq. (1) in the form:

$$u \quad U(\xi), \quad \xi \quad x \quad dt \tag{2}$$

where d is a arbitrary constant. Using the wave variable (2), the eq. (1) is carried to:

$$a_0 U \quad dU \quad a_1 UU \quad a_2 U^{(3)} \quad b_0 U \quad b_1 (U^{\ 2} \quad UU\ ) \quad b_2 U^{(4)} \quad 0 \tag{3}$$

integrating eq. (3) once with respect to ξ and setting the integration constant as zero, we have:

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$$a_0 U \quad dU \quad \frac{a_1}{2} U^2 \quad b_0 U \quad b_1 U U \quad a_2 U \quad b_2 U^{(3)} \quad 0 \tag{4}$$

suppose that the solution of the ordinary differential equation (ODE) (4) can be expressed:

$$U(\xi) \quad \overset{m}{\underset{i \ 0}{}} c_i \ \frac{G}{G}^{\ i} \tag{5}$$

and $G(\xi)$ satisfies a second order linear ODE:

$$G \quad \lambda G \quad \mu G \quad 0 \tag{6}$$

where c_i are constants to be determined and $c_m \quad 0$, λ and μ are arbitrary constants, m can be determined by considering the homogeneous balance between the highest order derivative $U^{(3)}$ and non-linear term UU' appearing in (4), $m + 3 = 2m + 1$, so that $m = 2$.

We then suppose that eq. (4) has the following solutions:

$$U(\xi) \quad c_0 \quad c_1 \ \frac{G}{G} \quad c_2 \ \frac{G}{G}^{\ 2}, \quad c_2 \quad 0 \tag{7}$$

where c_2, c_1, and c_0 are constants to be determined.

Substituting eq. (7) along with eq. (6) into eq. (4) and collecting all the terms with the same power of G'/G together, equating each coefficient to zero, yields a set of algebraic equations, which is large and difficult to solve, with the aid of MATHEMATICA and the Wu elimination method [8], we can distinguish the different cases namely:

Case (1)

$$c_0 \quad \frac{2(\lambda^2 a_2 \quad 2\mu a_2)}{a_1}, \ c_2 \quad \frac{12 a_2}{a_1}, \ c_1 \quad \frac{12\lambda a_2}{a_1}, \qquad d \quad a_0 \quad \lambda^2 a_2 \quad 4\mu a_2, \ b_0 \quad b_1 \quad b_2 \quad 0 \tag{8}$$

Case (2)

$$c_2 \quad \frac{12 a_2}{a_1}, \ c_1 \quad \frac{12\lambda a_2}{a_1}, \ c_0 \quad \frac{12\mu a_2}{a_1}, \ d \quad a_0 \quad \lambda^2 a_2 \quad 4\mu a_2, \ b_0 \quad b_1 \quad b_2 \quad 0 \tag{9}$$

Case (3)

$$c_2 \quad \frac{12 b_2}{b_1}, \ c_1 \quad \frac{12(a_2 b_1 \quad a_1 b_2 \quad 5\lambda b_1 b_2)}{5 b_1^2}$$

$$c_0 \quad \frac{6(5 a_2^2 b_1 \quad 5 a_1 a_2 b_2 \quad 19\lambda a_2 b_1 b_2 \quad 5 b_0 b_1 b_2 \quad 19\lambda a_1 b_2^2 \quad 190\mu b_1 b_2^2)}{95 b_1^2 b_2} \tag{10}$$

$$d \quad \frac{36 a_2^3 b_1 \quad 36 a_1 a_2^2 b_2 \quad 150 a_2 b_0 b_1 b_2 \quad 114 a_1 b_0 b_2^2 \quad 361 a_0 b_1 b_2^2}{361 b_1 b_2^2}$$

and

$$b_1 a_2^2 \quad 76 b_1 b_2^2 \mu \quad 19 b_1 b_2^2 \lambda^2 \quad a_2 b_2 a_1 \quad b_2 b_1 b_0 \quad 0,$$

$$25 b_1^2 b_0 b_2 \quad 6 b_1^2 a_2^2 \quad 13 a_2 b_1 b_2 a_1 \quad 19 b_2^2 a_1^2 \quad 0$$

Case (4)

$$c_2 \quad \frac{12b_2}{b_1},\; c_1 \quad \frac{12(a_2 b_1 \quad a_1 b_2 \quad 5\lambda b_1 b_2)}{5b_1^2},$$

$$c_0 \quad \frac{6(5a_2^2 b_1 \quad 5a_1 a_2 b_2 \quad 31\lambda a_2 b_1 b_2 \quad 5b_0 b_1 b_2 \quad 31\lambda a_1 b_2^2 \quad 310\mu b_1 b_2^2)}{155 b_1^2 b_2}$$

$$d \quad \frac{36a_2^3 b_1 \quad 36a_1 a_2^2 b_2 \quad 150a_2 b_0 b_1 b_2 \quad 186a_1 b_0 b_2^2 \quad 961a_0 b_1 b_2^2}{961 b_1 b_2^2}, \tag{11}$$

$$b_1 a_2^2 \quad 124 b_1 b_2^2 \mu \quad 31 b_1 b_2^2 \lambda^2 \quad a_2 b_2 a_1 \quad b_2 b_1 b_0 \quad 0,$$

$$25b_1^2 b_0 b_2 \quad 6b_1^2 a_2^2 \quad 37a_2 b_1 b_2 a_1 \quad 31 b_2^2 a_1^2 \quad 0$$

Case (5)

$$c_0 \quad \frac{a_0 \quad a_2^3 \quad 6\lambda a_2^2 b_2 \quad 6a_2 b_0 b_2 \quad 36a_0 b_2^2 \quad 6\lambda^2 a_2 b_2^2 \quad 48\mu a_2 b_2^2}{6a_2 b_1 b_2}$$

$$c_2 \quad \frac{12b_2}{b_1},\; c_1 \quad \frac{2(a_2 \quad 6\lambda b_2)}{b_1},$$

$$d \quad \frac{a_0 \quad a_2^3 \quad 6a_2 b_0 b_2 \quad 6\lambda^2 a_2 b_2^2 \quad 24\mu a_2 b_2^2}{36 b_2^2}, \tag{12}$$

and

$$a_2 b_1 \quad 6a_1 b_2 \quad 0,$$

$$a_0^2 (1 \quad 36b_2^2)^2 \quad 4a_0 a_2 (1 \quad 36b_2^2)\{2a_2^2 \quad 3b_2 [b_0 \quad (\lambda^2 \quad 4\mu) b_2]\}$$

$$a_2^2 \{a_2^4 \quad 36b_2^2 [b_0 \quad (\lambda^2 \quad 4\mu) b_2]^2 \quad 12a_2^2 b_2 [b_0 \quad 4(\lambda^2 \quad 4\mu) b_2]\} \quad 0$$

Case (6)

$$c_2 \quad \frac{12a_2}{a_1},\; c_1 \quad \frac{12(5\lambda a_2 \quad b_0)}{5a_1},\; c_0 \quad \frac{6(50\mu a_2^2 \quad 5\lambda a_2 b_0 \quad b_0^2)}{25 a_1 a_2},$$

$$d \quad \frac{25a_0 a_2 \quad 6b_0^2}{25a_2},\text{ and }\; 100a_2^2 \mu \quad 25a_2^2 \lambda^2 \quad b_0^2 \quad 0,\; b_1 \quad 0,\; b_2 \quad 0 \tag{13}$$

Case (7)

$$c_2 \quad \frac{12a_2}{a_1},\; c_1 \quad \frac{12(5\lambda a_2 \quad b_0)}{5a_1},\; c_0 \quad \frac{6(50\mu a_2^2 \quad 5\lambda a_2 b_0 \quad b_0^2)}{25 a_1 a_2},$$

$$d \quad \frac{25a_0 a_2 \quad 6b_0^2}{25a_2},\text{ and }\; 100a_2^2 \mu \quad 25a_2^2 \lambda^2 \quad b_0^2 \quad 0,\; b_1 \quad b_2 \quad 0 \tag{14}$$

Case (8)

$$c_2 \quad \frac{12b_2}{b_1},\; c_1 \quad \frac{12(a_2 \quad 5\lambda b_2)}{5b_1},\; c_0 \quad \frac{6\lambda a_2 - 5b_0 \quad 60\mu b_2}{5b_1},$$

$$d \quad a_0,\text{ and }\; a_2^2 \quad 100b_2^2 \mu \quad 25b_2^2 \lambda^2 \quad 0,\; a_1 \quad 0 \tag{15}$$

Case (9)

$$c_2 \quad \frac{12b_2}{b_1},\; c_1 \quad \frac{12\lambda b_2}{b_1},\; c_0 \quad \frac{b_0 \quad \lambda^2 b_2 \quad 8\mu b_2}{b_1},\; d \quad a_0,\; a_1 \quad a_2 \quad 0 \tag{16}$$

Using the previous results, we can obtain nine families of travelling wave solutions, we just give two families of these solutions due to the limit of length, other families of solutions can be obtained in the same way.

Consider *Case (1)*, substituting (8) into (7), we obtain:

$$U(\xi) \quad \frac{2(\lambda^2 a_2 \quad 2\mu a_2)}{a_1} \quad \frac{12\lambda a_2}{a_1} \ \frac{G}{G} \quad \frac{12 a_2}{a_1} \ \frac{G}{G}^{\ 2}, \qquad \xi \quad x \quad (a_0 \quad \lambda^2 a_2 \quad 4\mu a_2)t \tag{17}$$

substituting the general solutions of eq. (6) into eq. (17), we can obtain solutions of eq. (1):

– When $\lambda^2 \quad 4\mu > 0$,

$$u(\xi) \quad \frac{12\lambda \quad \frac{\lambda}{2} \quad \frac{\sqrt{\lambda^2 \quad 4\mu}\ c2\cosh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c1\sinh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}{2\ c1\cosh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c2\sinh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}} \quad a_2}{a_1} \quad \frac{12 \quad \frac{\lambda}{2} \quad \frac{\sqrt{\lambda^2 \quad 4\mu}\ c2\cosh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c1\sinh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}{2\ c1\cosh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c2\sinh\ \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}^{\ 2} \quad a_2}{a_1} \quad \frac{2(\lambda^2 a_2 \quad 2\mu a_2)}{a_1}$$

where $\xi = x - (a_0 - \lambda^2 a_2 + 4\mu a_2)t$ and c1 and c2 are arbitrary constants.

– When $\lambda^2 - 4\mu < 0$,

$$u(\xi) \quad \frac{12\lambda \quad \frac{\lambda}{2} \quad \frac{\sqrt{-\lambda^2 \quad 4\mu}\ c2\cos\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c1\sin\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}}{2\ c1\cos\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c2\sin\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}} \quad a_2}{a_1} \quad \frac{12 \quad \frac{\lambda}{2} \quad \frac{\sqrt{\ \lambda^2 \quad 4\mu}\ c2\cos\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c1\sin\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}}{2\ c1\cos\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c2\sin\ \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}}^{\ 2} \quad a_2}{a_1} \quad \frac{2(\lambda^2 a_2 \quad 2\mu a_2)}{a_1}$$

where $\xi = x - (a_0 - \lambda^2 a_2 + 4\mu a_2)t$ and c1and c2 are arbitrary constants.

– When $\lambda^2 - 4\mu = 0$,

$$u(\xi) \quad \frac{6\lambda[c1\lambda \quad c2(\; 2 \quad \xi\lambda)]a_2}{(c1 \quad c2\xi)a_1} \quad \frac{3[c1\lambda \quad c2(\; 2 \quad \xi\lambda)]^2 a_2}{(c1 \quad c2\xi)^2 a_1} \quad \frac{2(\lambda^2 a_2 \quad 2\mu a_2)}{a_1}$$

where $\xi = x - (a_0 - \lambda^2 a_2 + 4\mu a_2)t$ and c1 and c2 are arbitrary constants.

Consider *Case (2)*, similar on *Case (1)*, we can obtain three types of traveling wave solutions of eq. (1):

– When $\lambda^2 - 4\mu > 0$,

$$u(\xi) \quad \frac{12\mu a_2}{a_1} \quad \frac{12\lambda \quad \frac{\lambda}{2} \quad \frac{\sqrt{\lambda^2 \quad 4\mu} \; c2\cosh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c1\sinh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}{2 \; c1\cosh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c2\sinh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}} \quad a_2}{a_1} \quad \frac{12 \quad \frac{\lambda}{2} \quad \frac{\sqrt{\lambda^2 \quad 4\mu} \; c2\cosh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c1\sinh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}{2 \; c1\cosh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu} \quad c2\sinh \frac{1}{2}\xi\sqrt{\lambda^2 \quad 4\mu}}^{\;2} \quad a_2}{a_1}$$

where $\xi = x - (a_0 + \lambda^2 a_2 - 4\mu a_2)t$ and c1 and c2 are arbitrary constants.

– When $\lambda^2 - 4\mu < 0$,

$$u(\xi) \quad \frac{12\mu a_2}{a_1} \quad \frac{12\lambda \quad \frac{\lambda}{2} \quad \frac{\sqrt{\;\lambda^2 \quad 4\mu} \; c2\cos \frac{1}{2}\xi\sqrt{\;\lambda^2 \quad 4\mu} \quad c1\sin \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}}{2 \; c1\cos \frac{1}{2}\xi\sqrt{\;\lambda^2 \quad 4\mu} \quad c2\sin \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}} \quad a_2}{a_1} \quad \frac{12 \quad \frac{\lambda}{2} \quad \frac{\sqrt{\;\lambda^2 \quad 4\mu} \; c2\cos \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c1\sin \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu}}{2 \; c1\cos \frac{1}{2}\xi\sqrt{-\lambda^2 \quad 4\mu} \quad c2\sin \frac{1}{2}\xi\sqrt{\;\lambda^2 \quad 4\mu}}^{\;2} \quad a_2}{a_1}$$

where $\xi = x - (a_0 + \lambda^2 a_2 - 4\mu a_2)t$ and c1and c2 are arbitrary constants.

– When $\lambda^2 - 4\mu = 0$,

$$u(\xi) \quad \frac{6\lambda[c1\lambda \quad c2(\; 2 \quad \xi\lambda)]a_2}{(c1 \quad c2\xi)a_1} \quad \frac{3[c1\lambda \quad c2(\; 2 \quad \xi\lambda)]^2 a_2}{(c1 \quad c2\xi)^2 a_1} \quad \frac{12\mu a_2}{a_1}$$

where $\xi = x - (a_0 + \lambda^2 a_2 - 4\mu a_2)t$ and c1and c2 are arbitrary constants.

Conclusion

In this paper, we use G'/G-expansion method to solve a wave equation arising in fluid mechanics, this process can be reduced to solve a large system of algebraic equations, which is hard to solve, then we use Wu elimination method to solve the algebraic equations. The results show that we get more and general solutions than [4, 9, 10].

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