CONSTRUCTION OF A VARIATIONAL FORMULATION FOR HEAT TRANSFER

by

Hong-Yan LIU [a,b*] ***and Juan-Fen JIANG*** [c]

[a] School of Fashion Technology, Zhongyuan University of Technology, Zhengzhou, China
[b] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China
[c] School of Arts, Jinling Institute of Technology, Nanjing, China

Original scientific paper
DOI: 10.2298/TSCI1603889L

This paper gives two useful methods to construct a variational formulation for heat transfer. Xu's variational principle for heat transfer in terms of entransy is used as an example to elucidate how to correct a wrong variational formulation, and the semi-inverse method is adopted to establish a needed variational principle.

Key words: *variational principle, Euler-Lagrange equation, semi-inverse method*

Introduction

In 2012, Xu [1] established a variational principle for heat transfer, which is:

$$J(T) \qquad \underset{V}{L\mathrm{d}V} \tag{1}$$

where L is the entransy defined [1]:

$$L \quad k \quad \frac{\partial T}{\partial x}^{2} \qquad \frac{\partial T}{\partial y}^{2} \qquad \frac{\partial T}{\partial z}^{2} \qquad kT \; \frac{\partial^2 T}{\partial x^2} \quad \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial^2 T}{\partial z^2} \tag{2}$$

The Euler-Lagrange equation of eq. (1) is:

$$\frac{\partial L}{\partial T} \quad \frac{\partial}{\partial x}\frac{\partial L}{\partial T_{x}} \qquad \frac{\partial}{\partial y}\frac{\partial L}{\partial T_{y}} \qquad \frac{\partial}{\partial z}\frac{\partial L}{\partial T_{z}} \qquad \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial T_{xx}}$$

$$\frac{\partial^2}{\partial y^2}\frac{\partial L}{\partial T_{yy}} \qquad \frac{\partial^2}{\partial z^2}\frac{\partial L}{\partial T_{zz}} \qquad \frac{\partial^2}{\partial xy}\frac{\partial L}{\partial T_{xy}} \qquad \frac{\partial^2}{\partial xz}\frac{\partial L}{\partial T_{xz}} \qquad \frac{\partial^2}{\partial yz}\frac{\partial L}{\partial T_{yz}} \qquad 0 \tag{3}$$

where the subscribes imply the partial differentiation, *i. e.* $T_x = \; T/ \; x$, $T_{xx} = \; ^2T/ \; x^2$.

By a simple calculation, we have:

* Corresponding author; e-mail: phdliuhongyan@yahoo.com

$$k \quad \frac{\partial^2 T}{\partial x^2} \quad \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial^2 T}{\partial z^2} \quad 2k \quad \frac{\partial}{\partial x} \quad \frac{\partial T}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial T}{\partial y} \quad \frac{\partial}{\partial z} \quad \frac{\partial T}{\partial z}$$

$$k \quad \frac{\partial^2}{\partial x^2}(T) \quad \frac{\partial^2}{\partial y^2}(T) \quad \frac{\partial}{\partial z^2}(T) \quad 0 \tag{4}$$

Equation (4) implies 0   0. We therefore, conclude that Xu's variational principle is wrong. In this paper we will show how to establish a needed variational formulation for practical applications.

Construction of a variational formulation

Try-and-error method

The Lagrange function given in eq. (2) gives a wrong result, so eq. (2) requires some modification. In this paper, we give the following correction:

$$L \quad ak \quad \frac{\partial T}{\partial x}^{2} \quad \frac{\partial T}{\partial y}^{2} \quad \frac{\partial T}{\partial z}^{2} \quad bkT \quad \frac{\partial^2 T}{\partial x^2} \quad \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial^2 T}{\partial z^2} \tag{5}$$

where a and b are constants.

The Euler-Lagrange equation of eq. (5) reads:

$$ak \quad \frac{\partial}{\partial x} \quad \frac{\partial T}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial T}{\partial y} \quad \frac{\partial}{\partial z} \quad \frac{\partial T}{\partial z} \quad bk \quad \frac{\partial^2}{\partial x^2}(T) \quad \frac{\partial^2}{\partial y^2}(T) \frac{\partial^2}{\partial z^2}(T) \quad 0 \tag{6}$$

or

$$(b \quad a)k \quad \frac{\partial^2 T}{\partial x^2} \quad \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial^2 T}{\partial z^2} \quad 0 \tag{7}$$

It requires that $a \quad b$.

Semi-inverse method

The semi-inverse method was proposed in 1997 to establish a variational formulation directly from governing equations [2]. Hereby we consider a general heat transfer:

$$\frac{\partial}{\partial x} \quad k_x \frac{\partial T}{\partial x} \quad \frac{\partial}{\partial y} \quad k_y \frac{\partial T}{\partial y} \quad \frac{\partial}{\partial z} \quad k_z \frac{\partial T}{\partial z} \quad Q \tag{8}$$

where k_x, k_y, k_z, and Q are constants or function of co-ordinates.

The semi-inverse method begins with a trial Lagrange function with an energy form. For example:

$$L \quad k_x \quad \frac{\partial T}{\partial x}^{2} \quad F \tag{9}$$

where F is an unknown function of T and/or its derivatives. Alternative approaches to construction of the Lagrange function was discussed in [3-6].

The Euler-Lagrange equation for eq. (9) is:

$$2\frac{\partial}{\partial x}\quad k_x\frac{\partial T}{\partial x}\qquad \frac{\delta F}{\delta T}\quad 0 \tag{10}$$

where $\delta F/\delta T$ is the variational derivative defined:

$$\frac{\delta F}{\delta T}\quad \frac{\partial F}{\partial T}\quad \frac{\partial}{\partial x}\ \frac{\partial F}{\partial T_x}\qquad \frac{\partial}{\partial y}\ \frac{\partial F}{\partial T_y}\qquad \frac{\partial}{\partial z}\ \frac{\partial F}{\partial T_z}\qquad \frac{\partial^2}{\partial x^2}\ \frac{\partial F}{\partial T_{xx}}$$

$$\frac{\partial^2}{\partial y^2}\ \frac{\partial F}{\partial T_{yy}}\qquad \frac{\partial^2}{\partial z^2}\ \frac{\partial F}{\partial T_{zz}}\qquad \frac{\partial^2}{\partial xy}\ \frac{\partial F}{\partial T_{xy}}\qquad \frac{\partial^2}{\partial xz}\ \frac{\partial F}{\partial T_{xz}}\qquad \frac{\partial^2}{\partial yz}\ \frac{\partial F}{\partial T_{yz}}\qquad \cdots \tag{11}$$

Equation (10) should satisfy eq. (8), this requires that:

$$\frac{\delta F}{\delta T}\quad 2\frac{\partial}{\partial x}\ k_x\frac{\partial T}{\partial x}\quad 2Q\quad 2\frac{\partial}{\partial y}\ k_y\frac{\partial T}{\partial y}\qquad \frac{\partial}{\partial z}\ k_z\frac{\partial T}{\partial z} \tag{12}$$

From eq. (12), F can be identified, which reads:

$$F\quad k_y\ \frac{\partial T}{\partial y}^{\ 2}\qquad k_z\ \frac{\partial T}{\partial z}^{\ 2}\qquad 2QT \tag{13}$$

We, therefore, obtain a needed Lagrange function:

$$L\quad k_x\ \frac{\partial T}{\partial x}^{\ 2}\qquad k_y\ \frac{\partial T}{\partial y}^{\ 2}\qquad k_z\ \frac{\partial T}{\partial z}^{\ 2}\qquad 2QT \tag{14}$$

Remark 1

In case $k_x = k_y = k_z = k$ and $Q = 0$, eq. (14) is equivalent to eq. (5) when $a = 1$ and $b = 0$.

Remark 2

When Q is a function of T, the Lagrange function becomes:

$$L\quad k_x\ \frac{\partial T}{\partial x}^{\ 2}\qquad k_y\ \frac{\partial T}{\partial y}^{\ 2}\qquad k_z\ \frac{\partial T}{\partial z}^{\ 2}\qquad 2QT \tag{15}$$

where Q is calculated from the equation:

$$Q\quad \frac{\partial q}{\partial T} \tag{16}$$

Conclusions

This paper shows how to correct a wrong variational formulation by a try-and-error method, and the semi-inverse method is adopted to establish a needed formulation, formulation directly from the governing equations.

Acknowledgment

The work is supported by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), Jiangsu Planned Projects for Postdoctoral Research Funds under grant No. 1401076B, China Postdoctoral Science Foundation under grant No. 2015M571806 and 2016T90495, China National Textile and Apparel Council Project under grant No. 2015011, and Key Scientific Reseatch Projects of Henan Province under grant No. 16A540001.

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Paper submitted: December 12, 2015
Paper revised: April 5, 2016
Paper accepted: June 12, 2016