ANALYTICAL APPROACH TO A GENERALIZED HIROTA-SATSUMA COUPLED KORTEWEG-DE VRIES EQUATION BY MODIFIED VARIATIONAL ITERATION METHOD

by

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In this paper, we apply the modified variational iteration method to a generalized Hirota-Satsuma coupled Korteweg-de Vries (KdV) equation. The numerical solutions of the initial value problem of the generalized Hirota-Satsuma coupled KdV equation are provided. Numerical results are given to show the efficiency of the modified variational iteration method.

Key words: *modified variational iteration method, generalized Hirota-Satsuma coupled KdV equation*

Introduction

The non-linear equations arise in many fields, such as nanoscale hydrodynamics, fluid mechanics, thermodynamics and others, which play an important role in the study of non-linear physical phenomena. In this paper, we consider a generalized Hirota-Satsuma coupled KdV equation:

$$u_t \quad \frac{1}{2}(u_{xxx} \quad 6uu_x) \quad 3(vw)_x, \; v_t \quad v_{xxx} \quad 3uv_x, \quad w_t \quad w_{xxx} \quad 3uw_x \tag{1}$$

When $w = v^*$ and $w = v$, eq. (1) reduces to a complex KdV equation [1] or the Hirota-Satsuma equation [2], respectively. In the passed decades, many numerical or analytical methods are proposed to solve eq. (1), such as variational iteration method (VIM) [3, 4], reduced form of differential transformation method [5], and homotopy analysis method [6]. The numerical simulation of the generalized Hirota-Satsuma coupled KdV equation helps to model the waves. In this paper, we will use a modified variational iteration method (MVIM) [7] for solving the initial value problems associated with eq. (1). Compared results of the numerical solutions and the exact solutions are presented, which shows that the MVIM is efficient for solving the generalized Hirota-Satsuma coupled KdV equation.

Modified variational iteration method

To illustrate the basic idea of MVIM, let us consider the following non-linear partial differential equation:

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = g(x, t)$$

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$$u(x, 0) = f(x) \tag{2}$$

where $L = \partial/\partial t$, R is a linear operator with the partial derivative with respect to x, $Nu(x, t)$ – a non-linear term, and $g(x, t)$ – an inhomogeneous term.

For speeding up the convergence and reducing the computation cost of VIM, the MVIM was proposed in [7]. The MVIM for eq. (2) is constructed by the following variational iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{R[u_n(x,\xi) - u_{n-1}(x,\xi)] + [G_n(x,\xi) - G_{n-1}(x,\xi)]\}\mathrm{d}\xi$$

with $u_{-1} = 0$, $u_0 = f(x)$, $u_1 = u_0 - \int_0^t [R(u_0 - u_{-1}) + (G_0 - G_{-1}) - g]\mathrm{d}\xi$, and $G_n(x, t)$ is given by $Nu_n(x, t) = G_n(x, t) + O(t^{n+1})$.

Applications and results

We consider the numerical solution of eq. (1) with the following initial conditions: $u(x, 0) = (1/3)(\beta - 2k^2) + 2k^2\tanh^2(kx)$, $v(x, 0) = (-4k^2c_0/3c_1^2)(\beta + k^2) + (4k^2/3c_1^2)(\beta + k^2)\tanh(kx)$, and $w(x, 0) = c_0 + c_1\tanh(kx)$, where k, c_0, $c_1 \neq 0$, and β are arbitrary constants.

By MVIM, it follows the iteration formulae:

$$\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left\{\frac{1}{2}[u_{nxxx}(x,\xi) - u_{n-1xxx}(x,\xi)] + [G_n(x,\xi) - G_{n-1}(x,\xi)]\right\}\mathrm{d}\xi \\
v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \{[v_{nxxx}(x,\xi) - v_{n-1xxx}(x,\xi)] + [H_n(x,\xi) - H_{n-1}(x,\xi)]\}\mathrm{d}\xi \\
w_{n+1}(x,t) &= w_n(x,t) - \int_0^t \{[w_{nxxx}(x,\xi) - w_{n-1xxx}(x,\xi)] + [Q_n(x,\xi) - Q_{n-1}(x,\xi)]\}\mathrm{d}\xi
\end{aligned} \tag{3}$$

where $G_n(x, t)$ is obtained from $3u_nu_{nx} - 3(v_nw_n)_x = G_n(x, t) + O(t^{n+1})$, and $H_n(x, t)$ is defined by $-3u_nv_{nx} = H_n(x, t) + O(t^{n+1})$, and $Q_n(x, t)$ is given by $-3u_nw_{nx} = G_n(x, t) + O(t^{n+1})$, u_{-1}, v_{-1}, w_{-1}, G_{-1}, H_{-1}, and Q_{-1} are set be zero.

We use the eq. (3) with the initial guesses $u_0 = u(x, 0)$, $v_0 = v(x, 0)$, and $w_0 = w(x, 0)$, and obtain the fourth order approximated solutions. We remark that the bell-type solution $u(x, t)$ and the kink-type solutions $v(x, t)$, $w(x, t)$ of eq. (1) are given by:

$$u(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2[k(x + \beta t)]$$

$$v(x,t) = -\frac{4k^2c_0}{3c_1^2}(\beta + k^2) + \frac{4k^2}{3c_1^2}(\beta + k^2)\tanh[k(x + \beta t)]$$

$$w(x,t) = c_0 + c_1\tanh[k(x + \beta t)]$$

respectively. In this example, $k = 0.2$, $\beta = 0.1$, $c_0 = 1.5$, and $c_1 = 0.1$.

Tables 1-3 list the relative errors of the MVIM solutions u_4, v_4, and w_4, respectively. Figure 1 plots the compared results of the MVIM solution u_4 and the bell-type solution $u(x, t)$ of eq. (1). The numerical results for the approximation v_4 and the kink-type solution $v(x, t)$ are shown in fig. 2. The numerical solution w_4 and the kink-type solution $w(x, t)$ are also plotted in fig. 3. Different from the VIM, the MVIM works well for this example. Particularly, MVIM solutions agree well with the exact solutions of eq. (1) when $-50 \le x, t \le 50$.

Table 1. Relative errors of MVIM solutions, u_4

T	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
1	$5.31 \cdot 10^{-9}$	$2.22 \cdot 10^{-10}$	$4.06 \cdot 10^{-10}$	$9.29 \cdot 10^{-11}$	$3.29 \cdot 10^{-12}$
2	$1.60 \cdot 10^{-7}$	$7.35 \cdot 10^{-9}$	$1.27 \cdot 10^{-8}$	$2.91 \cdot 10^{-9}$	$1.14 \cdot 10^{-10}$
3	$1.14 \cdot 10^{-6}$	$5.75 \cdot 10^{-8}$	$9.50 \cdot 10^{-8}$	$2.16 \cdot 10^{-8}$	$9.27 \cdot 10^{-10}$
4	$4.54 \cdot 10^{-6}$	$2.49 \cdot 10^{-7}$	$3.93 \cdot 10^{-7}$	$8.93 \cdot 10^{-8}$	$4.16 \cdot 10^{-9}$
5	$1.31 \cdot 10^{-5}$	$7.76 \cdot 10^{-7}$	$1.18 \cdot 10^{-6}$	$2.67 \cdot 10^{-7}$	$1.35 \cdot 10^{-8}$

Table 2. Relative errors of MVIM solutions, v_4

T	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
1	$1.55 \cdot 10^{-12}$	$1.41 \cdot 10^{-11}$	$5.35 \cdot 10^{-12}$	$2.08 \cdot 10^{-13}$	$1.09 \cdot 10^{-12}$
2	$4.03 \cdot 10^{-11}$	$4.51 \cdot 10^{-10}$	$1.69 \cdot 10^{-10}$	$7.29 \cdot 10^{-12}$	$3.48 \cdot 10^{-11}$
3	$2.37 \cdot 10^{-10}$	$3.42 \cdot 10^{-9}$	$1.26 \cdot 10^{-9}$	$6.02 \cdot 10^{-11}$	$2.64 \cdot 10^{-10}$
4	$7.10 \cdot 10^{-10}$	$1.44 \cdot 10^{-8}$	$5.25 \cdot 10^{-9}$	$2.73 \cdot 10^{-10}$	$1.11 \cdot 10^{-9}$
5	$1.30 \cdot 10^{-9}$	$4.38 \cdot 10^{-8}$	$1.58 \cdot 10^{-8}$	$8.91 \cdot 10^{-10}$	$3.39 \cdot 10^{-9}$

Table 3. Relative errors of MVIM solutions, w_4

T	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
1	$1.47 \cdot 10^{-12}$	$1.29 \cdot 10^{-11}$	$4.78 \cdot 10^{-12}$	$1.83 \cdot 10^{-13}$	$9.58 \cdot 10^{-13}$
2	$3.82 \cdot 10^{-11}$	$4.11 \cdot 10^{-10}$	$1.51 \cdot 10^{-10}$	$6.44 \cdot 10^{-12}$	$3.06 \cdot 10^{-11}$
3	$2.24 \cdot 10^{-10}$	$3.11 \cdot 10^{-9}$	$1.13 \cdot 10^{-9}$	$5.31 \cdot 10^{-11}$	$2.32 \cdot 10^{-10}$
4	$6.69 \cdot 10^{-10}$	$1.31 \cdot 10^{-8}$	$4.69 \cdot 10^{-9}$	$2.41 \cdot 10^{-10}$	$9.78 \cdot 10^{-10}$
5	$1.22 \cdot 10^{-9}$	$3.98 \cdot 10^{-8}$	$1.41 \cdot 10^{-8}$	$7.87 \cdot 10^{-10}$	$2.98 \cdot 10^{-9}$

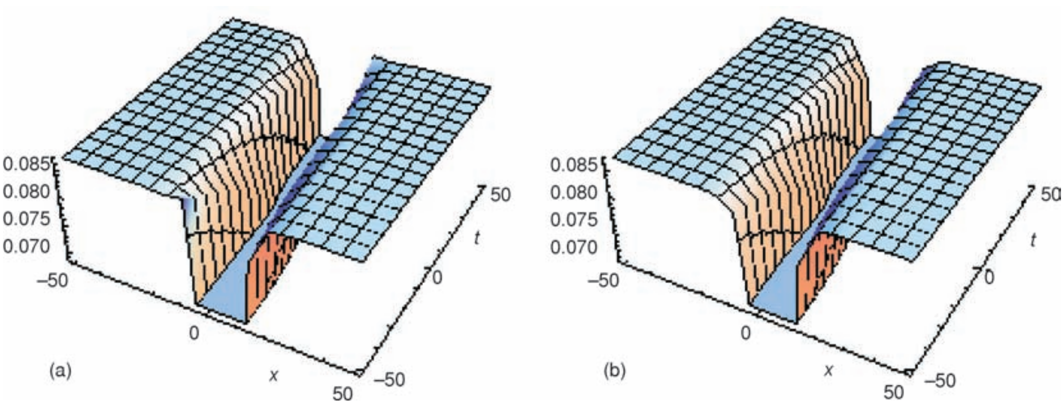

Figure 1. The compared results for the approximation, u_4 (a) and the exact solution, u (b)

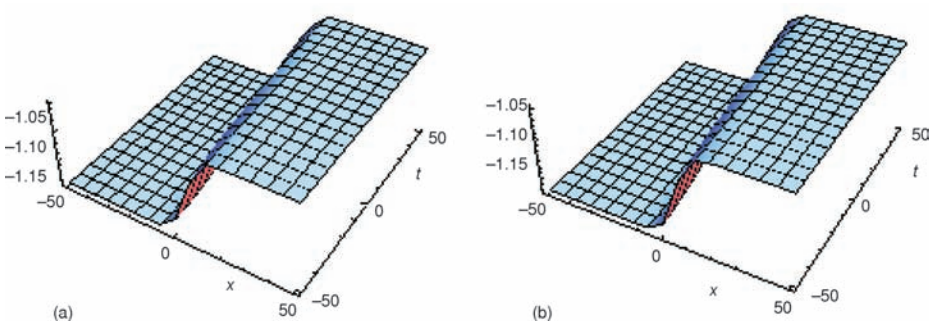

Figure 2. The numerical results for the approximation, v_4 (a) and the exact solution, *v* (b)

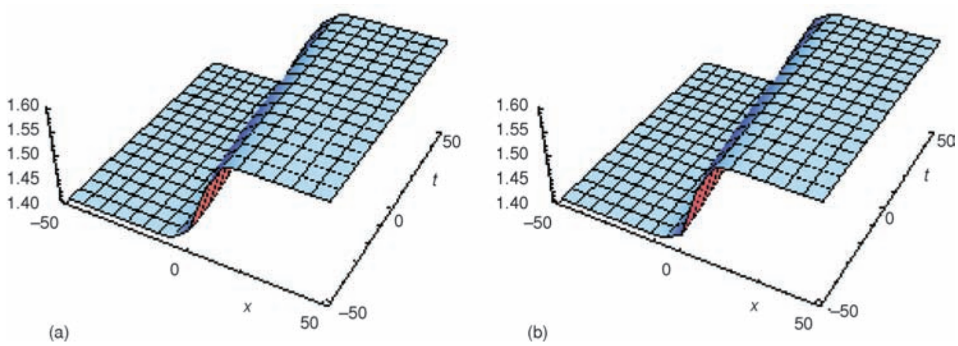

Figure 3. The approximation, w_4 (a) and the exact solution, *w* (b) when –50   *x* *t*   50

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