A DIRECT ALGORITHM OF EXP-FUNCTION METHOD FOR NON-LINEAR EVOLUTION EQUATIONS IN FLUIDS

by

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In this paper, a direct algorithm of the exp-function method is proposed for exactly solving non-linear evolution equations. To illustrate the validity and advantages of the algorithm, the Korteweg-de Vries and Jimbo-Miwa equations are considered. As a result, exact solutions are obtained. It is shown that the exp-function method with the direct algorithm provides a simpler but effective mathematical tool for constructing exact solutions of non-linear evolution equations in fluids.

Key words: *exp-function method, exact solution, Korteweg-de Vries equation, Jimbo-Miwa equation*

Introduction

Searching for exact solutions of non-linear evolution equations (NLEE) plays an important role in the study of non-linear physical phenomena in fluid dynamics. In 2006, He and Wu [1] proposed the so-called exp-function method to solve non-linear wave equations. It is shown that the exp-function method and its improvements are available for many NLEE such as those in [2-10]. This is due to the method possesses a more general ansatz:

$$u = \frac{\sum_{n=-f}^{g} a_n \mathrm{e}^{n\xi}}{\sum_{m=-p}^{q} b_m \mathrm{e}^{m\xi}}, \quad \xi = \sum_{i=1}^{s} k_i x_i - wt \tag{1}$$

supposed for a given NLEE with independent variables $t, x_1, x_2, ..., x_s$ and dependent variable u:

$$F(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_m}, u_{x_1 t}, u_{x_2 t}, \ldots, u_{x_s t}, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots, u_{x_s x_s}, \ldots) = 0 \tag{2}$$

Here a_n and b_m are undetermined constants, f, p, g, and q can be determined by balancing the highest order linear term with the highest order non-linear term in eq. (2). In general, the final solution does not strongly depend on the choices of values of f, p, g, q, and $f = p = g = q = 1$ is the simplest choice. In this paper, we would like to introduce the following general approach:

$$u = \sum_{j=1}^{n} u_j(t, x_1, x_2, \ldots, x_s)\varphi^{j-n}, \quad \varphi = 1 + \mathrm{e}^{\xi} \tag{3}$$

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where non-negative integer n is determined by balancing the highest order linear term with the highest order non-linear term in eq. (2), $u_j = u_j(t, x_1, x_2, ..., x_s), j = 1, 2,..., s$, are coefficient functions to be determined by the system of partial differential equations resulted from substituting eq. (3) into eq. (2). In the next section, the Korteweg-de Vries equation [11]:

$$u_t \quad 6uu_x \quad u_{xxx} \quad 0 \tag{4}$$

and the Jimbo-Miwa equation [12]:

$$u_{xxxy} \quad 3u_y u_{xx} \quad 3u_x u_{xy} \quad 2u_{yt} \quad 3u_{xz} \quad 0 \tag{5}$$

are taken as two examples to illustrate the steps of our algorithm.

Example 1

For the Korteweg-de Vries eq. (4), we balance the highest order linear term u_{xxx} with the highest order non-linear term $6uu_x$ and then have $n + 3 = n + n + 1$, *i. e.*, $n = 2$. In this case, eq. (3) becomes:

$$u \quad u_0(x,t)\varphi^{2} \quad u_1(x,t)\varphi^{1} \quad u_2(x,t), \quad \varphi \quad 1 \quad \mathrm{e}^{\xi}, \quad \xi \quad kx \quad ct \tag{6}$$

Substituting eq. (6) into eq. (4) and then equating each coefficient of the same order power of $\varphi^i (i = 0, -1, -2, ... , -5)$ to zero yields a set of partial differential equations:

$$\varphi^{5}: \quad 24k^3\mathrm{e}^{3\xi}u_0 \quad 12k\mathrm{e}^{\xi}u_0^2 \quad 0 \tag{7}$$

$$\varphi^{4}: 18k^3\mathrm{e}^{2\xi}u_0 \quad 6k^3\mathrm{e}^{3\xi}u_1 \quad 18k\mathrm{e}^{\xi}u_0u_1 \quad 18k^2\mathrm{e}^{2\xi}u_{0x} \quad 6u_0u_{0x} \quad 0 \tag{8}$$

$$\begin{gathered}\varphi^{3}: \quad 2c\mathrm{e}^{\xi}u_0 \quad 2k^3\mathrm{e}^{\xi}u_0 \quad 6k^3\mathrm{e}^{2\xi}u_1 \\ 6k\mathrm{e}^{\xi}u_1^2 \quad 12k\mathrm{e}^{\xi}u_0u_2 \quad 6k^2\mathrm{e}^{\xi}u_{0x} \quad 6u_0u_{1x} \quad 6k\mathrm{e}^{\xi}u_{0xx} \quad 0\end{gathered} \tag{9}$$

$$\begin{gathered}\varphi^{2}: \quad c\mathrm{e}^{\xi}u_1 \quad k^3\mathrm{e}^{\xi}u_1 \quad 6k\mathrm{e}^{\xi}u_1u_2 \quad u_{0x} \quad 6u_2u_{0x} \\ 3k^2\mathrm{e}^{\xi}u_{1x} \quad 6u_1u_{1x} \quad 6u_0u_{2x} \quad 3k\mathrm{e}^{\xi}u_{1xx} \quad u_{0xxx} \quad 0\end{gathered} \tag{10}$$

$$\varphi^{1}: u_{1t} \quad 6u_2u_{1x} \quad 6u_1u_{2x} \quad u_{1xxx} \quad 0 \tag{11}$$

$$\varphi^{0}: u_{2t} \quad 6u_2u_{2x} \quad u_{2xxx} \quad 0 \tag{12}$$

Solving eqs. (7)-(12), we have:

$$u_0 \quad 2k^2\mathrm{e}^{2\xi}, \; u_1 \quad 2k^2\mathrm{e}^{\xi}, \; u_2 \quad \frac{k^3 \quad c}{6k} \tag{13}$$

and hence obtain an exact solution of the Korteweg-de Vries eq. (4):

$$u \quad \frac{2k^2\mathrm{e}^{2\xi}}{(1 \quad e^{\xi})^2} \quad \frac{2k^2\mathrm{e}^{\xi}}{1 \quad \mathrm{e}^{\xi}} \quad \frac{k^3 \quad c}{6k} \quad \frac{1}{2}k^2 \,\mathrm{sech}^2 \frac{1}{2}\xi \quad \frac{k^3 \quad c}{6k}, \; \xi \quad kx \quad ct \tag{14}$$

Example 2

Let us consider the Jimbo-Miwa eq. (5). To begin with, we balance the highest order linear term u_{xxxy} with the highest order non-linear term $3u_yu_{xx}$ and then have $n + 4 = 2n + 3$, *i. e.*, $n = 1$. In this case, eq. (3) becomes:

$$u = u_0(x, y, z, t)\varphi^{-1} + u_1(x, y, z, t), \;\; \varphi = 1 + \mathrm{e}^{\xi}, \quad = kx + w(y, z, t) \tag{15}$$

Substituting eq. (15) into eq. (5) and then equating each coefficient of the same order power of $\varphi^i (i = 0, -1, -2, ... , -5)$ to zero yields a set of partial differential equations:

$$\varphi^{5}: 24k^3 e^{4\xi} w_{1y} u_0 \quad 12k^2 e^{3\xi} w_{1y} u_0 \quad 0 \tag{16}$$

$$\varphi^{4}: \quad 36k^3 e^{3\xi} w_y u_0 \quad 6k^2 e^{2\xi} w_y u_0^2 \quad 6k^3 e^{3\xi} u_{0y} \quad 9k^2 e^{2\xi} u_0 u_{0y} \\ 18k^2 e^{3\xi} w_y u_{0y} \quad 15ke^{2\xi} w_y u_0 u_{0x} \quad 0 \tag{17}$$

$$\varphi^{3}: \quad 6ke^{2\xi} w_2 u_0 \quad 14k^3 e^{2\xi} w_3 u_0 \quad 4e^{2\xi} w_y w_z u_0 \quad 6k^3 e^{2\xi} u_{0y} \quad 3k^2 e^{\xi} u_0 u_{0y} \quad 6k^2 e^{2\xi} u_0 u_{1y} \\ 18k^2 e^{2\xi} w_y u_{0x} \quad 3ke^{\xi} w_y u_0 u_{0x} \quad 9ke^{\xi} u_{0x} u_{0y} \quad 3e^{\xi} w_y u_{0x}^2 \quad 6ke^{2\xi} w_y u_0 u_{1x} \\ 6k^2 e^{2\xi} u_{0xy} \quad 3ke^{\xi} u_0 u_{0xy} \quad 6ke^{2\xi} w_y u_{0xx} \quad 3e^{\xi} w_y u_0 u_{0xx} \quad 0 \tag{18}$$

$$\varphi^{2}: 3ke^{\xi} w_z u_0 \quad k^3 e^{\xi} w_y u_0 \quad 2e^{\xi} w_y w_t u_0 \quad 2e^{\xi} w_{yt} u_0 \quad 2e^{\xi} w_y u_{0t} \quad 3ke^{\xi} u_{0z} \quad k^3 e^{\xi} u_{0y} \\ 2e^{\xi} w_t u_{0y} \quad 3k^2 e^{\xi} u_0 u_{1y} \quad 3e^{\xi} w_z u_{0x} \quad 3k^2 e^{\xi} w_y u_{0x} \quad 6ke^{\xi} u_{1y} u_{0x} \quad 3ke^{\xi} w_y u_0 u_{1x} \\ 3ke^{\xi} u_{0y} u_{1x} \quad 3e^{\xi} w_y u_{0x} u_{1x} \quad 3k^2 e^{\xi} u_{0xy} \quad 3u_{0x} u_{0xy} \quad 3ke^{\xi} u_0 u_{1xy} \quad 3ke^{\xi} w_y c_{0xx} \\ 3u_{0y} u_{0xx} \quad 3e^{\xi} w_y u_0 u_{1xx} \quad 3ke^{\xi} u_{0xxy} \quad e^{\xi} w_y u_{0xxx} \quad 0 \tag{19}$$

$$\varphi^{1}: 2u_{0yt} \quad 3u_{0xz} \quad 3u_{1x} u_{0xy} \quad 3u_{0x} u_{1xy} \quad 3u_{1y} u_{0xx} \quad 3u_{0y} u_{1xx} \quad u_{0xxxy} \quad 0 \tag{20}$$

$$\varphi^{0}: 2u_{1yt} \quad 3u_{1xz} \quad 3u_{1x} u_{1xy} \quad 3u_{1x} u_{1xx} \quad u_{1xxxy} \quad 0 \tag{21}$$

From eqs. (16)-(21), we have:

$$u_0 \quad 2ke^{\xi}, \quad u_1 \quad \frac{1}{3k^2} \quad \{3k[f_3(z) \quad f_1(y,z)] \quad [k^3 \quad 2p] f_{1y}(y,z)\} \mathrm{d}y \tag{22}$$

$$w \quad f_1(y,z) \quad f_2(z) \quad pt \tag{23}$$

where $f_1(y, z)$ and $f_2(z)$ are arbitrary functions of the indicated variables, $f_3(z) \quad \mathrm{d}f_3(z)/\mathrm{d}z$ and p is an arbitrary constant.

We therefore obtain an exact solution of the Jimbo-Miwa equation (5):

$$u \quad \frac{2ke^{\xi}}{1 \quad e^{\xi}} \quad \frac{1}{3k^2} \quad \{3k[f_3(z) \quad f_1(y,z)] \quad [k^3 \quad 2p] f_{1y}(y,z)\} \mathrm{d}y \tag{24}$$

where $\xi = kx + f_1(y, z) + f_2(z) + pt$.

It is shown from computer running that the general approach (3) has an advantage over the one in eq. (1) to deal with the so-called middle expression expansion problem appeared in the process of using the exp-function method to transform a given non-linear evolution equation into the over determined system of algebraic or differential equations.

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