RATIONAL SOLUTION TO A SHALLOW WATER WAVE-LIKE EQUATION

by

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Two classes of rational solutions to a shallow water wave-like non-linear differential equation are constructed. The basic object is a generalized bilinear differential equation based on a prime number, p = 3. Through this new transformation and with the help of symbolic computation with MAPLE, both the new equation and its rational solutions are obtained.

Key words: *rational solution, generalized bilinear equation, shallow water wave-like equation*

Introduction

In recent years, there has been a growing interest in non-linear differential equations, which are used to describe the mechanical, process control, ecological and economic systems, chemical re-cycling system and other areas of epidemiology issues [1, 2].

Shallow water wave equation is a mathematical description of a wide variety of shallow water fluid motion [3, 4]. In the research of shallow water equations, how to get the rational solutions is very important, however, the difficulty increases due to its non-linearity. If more exact solutions can be obtained through a simple way, a wide application are predicted [5-7].

In this paper, we would like to consider a shallow water wave-like non-linear differential equation induced from a generalized bilinear differential equation of shallow water wave type [8, 9]. Based on the original shallow water wave equation, a new bilinear transformation is considered. Then through the dependent variable transformation, we get the shallow water wave-like equation [10, 11]. From a class of polynomial generating functions, a MAPLE search tells us six classes of rational solutions to the considered shallow water wave-like equation, along with some special interesting solutions. A conjecture on rational solutions to the considered shallow water wave-like equation is made at the end of the paper.

A shallow water wave-like equation

Let us consider a generalized bilinear differential equation of shallow water wave-like type:

$$(D_x^3 D_t \quad D_x^2 \quad D_x D_t) ff \quad 6 f_{xx} f_{xt} \quad 2 f_{xx} f \quad 2 f_x^2 \quad 2 f_{xt} f \quad 2 f_x f_t \quad 0 \tag{1}$$

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This is the same type bilinear equation as the shallow water wave one [12]. The previous differential operators are some kind of generalized bilinear differential operators introduced in [13]:

$$D^m_{p,x}D^n_{p,t}ff = \left(\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'}\right)^n f(x,t)f(x',t')\Big|_{x'=x,t'=t} \tag{2}$$

where $\alpha_p^s = (-1)^{r_p(s)}$, $s = r_p(s) \bmod p$.

In particular, we have:

$$\alpha_3 = -1,\ \alpha_3^2 = 1,\ \alpha_3^3 = 1,\ \alpha_3^4 = -1,\ \alpha_3^5 = 1,\ \alpha_3^6 = 1 \tag{3}$$

and thus

$$D^3_{3,x}D_{3,t}ff = 6f_{xx}f_{xt},\ D^2_{3,x}ff = 2f_{xx}f - 2f_x^2,\ D_{3,x}D_{3,t}ff = 2f_{xt}f - 2f_xf_t \tag{4}$$

In the case of $p = 2$, *i. e.*, the Hirota case, we have:

$$\begin{aligned} &D^3_{2,x}D_{2,t}ff = 2f_{xxxt}f + 6f_{xx}f_{xt} - 6f_{xxt}f_x - 2f_{xxx}f_t \\ &D^2_{2,x}ff = 2f_{xx}f - 2f_x^2,\ D_{2,x}D_{2,t}ff = 2f_{xt}f - 2f_xf_t \end{aligned} \tag{5}$$

which generates the standard bilinear [14] shallow water wave equation. Under the link between f and u [15, 16]:

$$u = (\ln f)_x \tag{6}$$

and then the result of these transformation can directly show that the generalized bilinear eq. (1) is linked to a shallow water wave-like scalar non-linear differential equation:

$$\frac{3}{8}\left(\int u_t\mathrm{d}x\right)u^3 + \frac{3}{4}\left(\int u_t\mathrm{d}x\right)uu_x + \frac{3}{4}u_tu^2 + \frac{3}{2}u_tu_x + u_t + u_x = 0 \tag{7}$$

Because of the new equation is result from the transformation based on the bilinear forms of the original shallow water wave equation, it is called the shallow water wave-like equation. More precisely, by virtue of the transformation (6), the following equality holds:

$$\frac{(D_x^3D_t + D_x^2 + D_xD_t)ff}{f^2} = \frac{3}{8}\left(\int u_t\mathrm{d}x\right)u^3 + \frac{3}{4}\left(\int u_t\mathrm{d}x\right)uu_x + \frac{3}{4}u_tu^2 + \frac{3}{2}u_tu_x + u_t + u_x \tag{8}$$

and thus, f solves eq. (1) if and only if $u = (\ln f)_x$ presents a solution to the shallow water wave-like eq. (7). Resonant solutions in term of exponential functions and trigonometric functions [17] have been considered for generalized bilinear equations. In this paper, we would like to discuss their polynomial solutions which generate rational solutions to scalar non-linear differential equations by focusing on the shallow water wave-like eq. (7).

Rational solutions

By symbolic computation with MAPLE, we look for polynomial solutions, with degrees of x and t being less than 10:

$$f = \sum_{i=0}^{10}\sum_{j=0}^{10} c_{ij}x^it^j \tag{9}$$

where the c_{ij} are constants, and find many classes of polynomial solutions to the generalized bilinear equation. Among these solutions, we selected six solutions which hold a relatively simple form into consideration:

$$f = \frac{c_{21}}{3}x^3 + \frac{c_{11}}{2}x^2 + \frac{c_{11}^2}{4c_{21}}t + 12c_{21}t + \frac{c_{11}^2}{4c_{21}}x + c_{00} \tag{10}$$

$$f \quad 3c_{30}tx^2 \quad c_{30}x^2 \quad c_{20}t^2 \quad 2c_{20}tx \quad c_{20}x^2 \quad 3t^2$$
$$c_{20}t^2 \quad \frac{c_{20}^2}{3c_{30}}t \quad 36c_{30}t \quad 3c_{30}t^2x \quad \frac{c_{20}^2}{3c_{30}}x \quad c_{00} \tag{11}$$

$$f \quad c_{30}t^3 \quad \frac{2}{9}c_{12}t^3 \quad \frac{1}{9}t^2c_{11} \quad \frac{1}{3}c_{11}tx \quad \frac{1}{3}c_{01}t \quad \frac{2}{3}c_{12}t^2x \tag{12}$$

$$f \quad c_{41}t^5 \quad c_{41}x^4t \quad 12c_{41}t^3 \quad 6c_{41}t^3x^2 \quad 36c_{41}tx^2$$
$$4c_{41}x^3t^2 \quad 4c_{41}t^4x \quad 24c_{41}t^2x \quad 108c_{41}t \tag{13}$$

$$f \quad \frac{1}{16}c_{42}t^6 \quad \frac{1}{2}c_{42}t^5x \quad 2c_{42}t^3x^3 \quad c_{42}t^2x^4 \quad 3c_{42}t^4$$
$$12c_{42}t^3x \quad 36c_{42}t^2x^2 \quad 108c_{42}t^2 \tag{14}$$

$$f \quad c_{21}t^3 \quad c_{21}xt^2 \quad c_{21}tx^2 \quad \frac{1}{3}c_{21}x^3 \quad 2c_{20}t^2$$
$$2c_{20}tx \quad c_{20}x^2 \quad \frac{c_{20}^2}{c_{21}}t \quad 12c_{21}t \quad \frac{c_{20}^2}{c_{21}}x \quad c_{00} \tag{15}$$

where the involved constants c_{ij} 's are arbitrary provided that the solutions make sense.

Taking the concrete forms of the resulting polynomial solutions (10)-(15) into consideration, we can obtain six classes of rational solutions to the shallow water wave-like eq. (7) with the help of MAPLE:

$$u \quad \frac{6(4a^2x \quad 4abx \quad b^2)}{4a^2x^3 \quad 6abx^2 \quad 144a^2t \quad 3b^2t \quad 3b^2x \quad 12ac} \tag{16}$$

$$u \quad \frac{18t^2 \quad 36tx \quad 18x^2 \quad 480t \quad 480x \quad 3200}{9t^3 \quad 9t^2x \quad 9tx^2 \quad 3x^3 \quad 240tx \quad 120x^2 \quad 1708t \quad 1600x} \tag{17}$$

$$u \quad \frac{16t^3 \quad 96t^2x \quad 192tx^2 \quad 128x^3 \quad 384t \quad 2304x}{t^4 \quad 8t^3x \quad 24t^2x^2 \quad 32tx^3 \quad 16x^4 \quad 48t^2 \quad 192tx \quad 576x^2 \quad 1728} \tag{18}$$

$$u \quad \frac{12at^2 \quad 6bt}{9ct^3 \quad 9at^2x \quad 6dt^2 \quad 3btx \quad 3et \quad 3axt^2 \quad 2at^3 \quad bt^2} \tag{19}$$

$$u \quad \frac{8t^3 \quad 24t^2x \quad 24tx^2 \quad 8x^3 \quad 48t \quad 144x}{t^4 \quad 4t^3x \quad 6t^2x^2 \quad 4tx^3 \quad x^4 \quad 12t^2 \quad 24tx \quad 36x^2 \quad 108} \tag{20}$$

$$u \quad \frac{6(a^2t^2 \quad 2a^2tx \quad a^2x^2 \quad 2abt \quad 2abx \quad b^2)}{3a^2t^3 \quad 3a^2t^2x \quad 3a^2tx^2 \quad a^2x^3 \quad 6abt^2 \quad 6abtx \quad 3abx^2 \quad 36a^2t \quad 3b^2t \quad 3b^2x \quad 3ac} \tag{21}$$

where a, b, c, d, and e are arbitrary constants. Actually, the polynomial solutions in the first group of (10)-(15) generate the rational solutions in (16)-(21). Note that in (16)-(21), the constants were rescaled and renamed. Pictures of the solution (17), (18), and (20) are given in fig. 1. The rational solutions (16), on one hand, reduced to:

$$u \quad \frac{6x^2 \quad 60x \quad 150}{x^3 \quad 15x^2 \quad 111t \quad 75x} \tag{22}$$

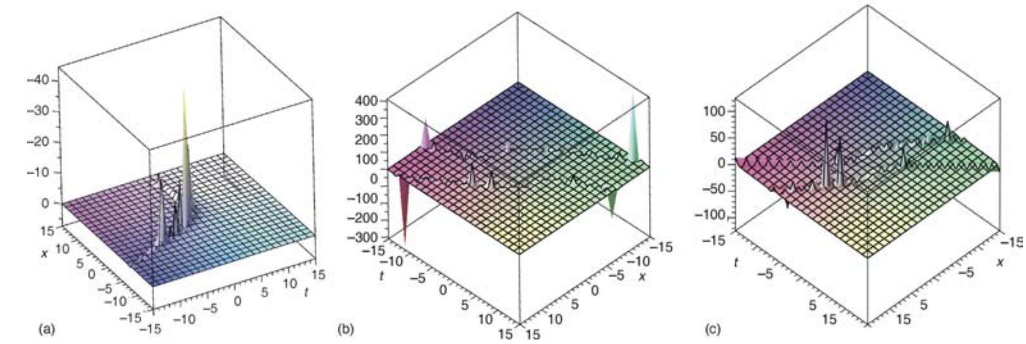

Figure 1. The rational solution: (a) eq. (17); (b) eq. (18); (c) eq. (20)
(for color image see journal web-site)

when a = 1, b = 10, and c = 0. On the other hand, the rational solutions (19) reduces to:

$$u \quad \frac{12t \quad 6}{11t^2 \quad 6tx \quad t \quad 3x} \tag{23}$$

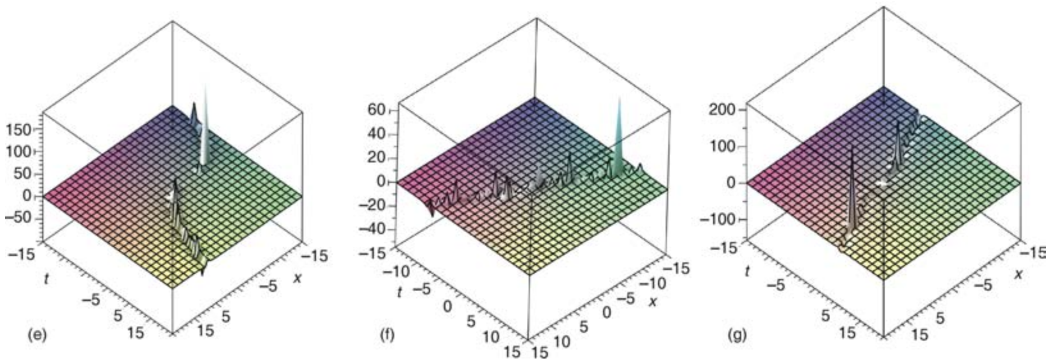

Figure 2. The rational solution: (e) eq. (22); (f) eq. (23); (g) eq. (24)
(for color image see journal web-site)

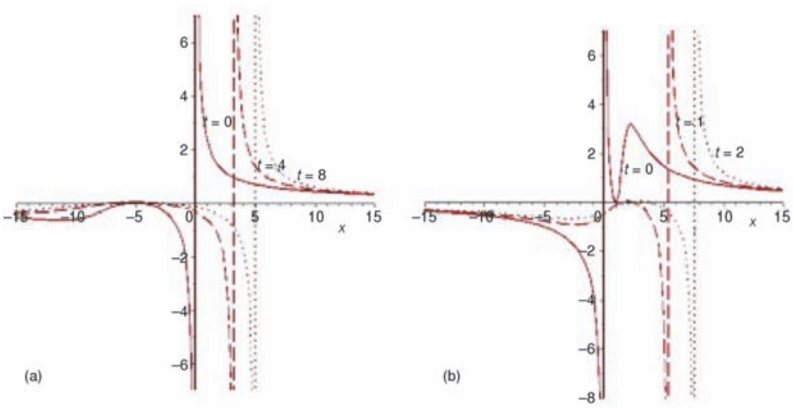

Figure 3. The *x*-curves of solutions (22) and (24): (a) eq. (22); (b) eq. (24)

when $a = 1$, $b = 1$, $c = 1$, $d = 0$, and $e = 0$. The rational solutions (21) reduces to:

$$u \quad \frac{6(t^2 \quad 2tx \quad x^2 \quad 2t \quad 2x \quad 1)}{3t^3 \quad 3t^2x \quad 3tx^2 \quad x^3 \quad 6t^2 \quad 6tx \quad 3x^2 \quad 39t \quad 3x} \tag{24}$$

when $a = 1$, $b = 1$, and $c = 0$. Pictures of the solution (22)-(24) are given in fig. 2. The x-curves of solutions (22) and (24) are depicted in fig. 3.

Conclusions

We considered a generalized bilinear equation which yields a shallow water wave--like equation [18], and constructed the rational solutions to the resulting shallow water wave-like eq. (7). By a MAPLE symbolic computation, we presented six classes of the constructed rational solutions. The basic starting point is a kind of generalized bilinear differential operators introduced in [19].

It is worth checking if there exists a kind of Wronskian solutions and lump solutions [20] to the shallow water wave-like non-linear eq. (7). We also conjecture that the six classes of rational solutions in (16)-(21) which generated from polynomial solutions to the generalized bilinear eq. (7) under the link (6) would contain all rational solutions to the shallow water wave-like nonlinear eq. (7).

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