RATIONAL SOLUTIONS TO AN CAUDREY-DODD-GIBBON-SAWADA-KOTERA-LIKE EQUATION

by

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This paper applies an improved Hirota bilinear differential operator to obtain a Caudrey-Dodd-Gibbon-Sawada-Kotera-like (CDGSK-like) equation, and two classes of rational solutions are obtained.

Key words: *CDGSK-like equation, Hirota bilinear operator, rational solutions*

Introduction

Non-linear partial differential equations (NLPDE) are attracting more and more attention [1-5] in thermal science and fluid mechanics, and it becomes more and more important to solve NLPDE exact solutions. Rational solutions to integrable equations have been considered systematically by using the Wronskian formulation and the Casoratian formulation [6]. Particular examples include the Korteweg-de Vries (KdV) equation, the Boussinesq equation, and the Toda lattice equation [7-9].

An CDGSK-like differential equation

In order to study the solution solutions of NLPDE, Hietarinta [10] introduces the following differential operator:

$$\mathrm{D}_t^m \mathrm{D}_x^n (fg) \quad (\partial_t \quad \partial_t \ )^m (\partial_x \quad \partial_x \ )^n [f(t,x)g(t \ ,x \ )]|_{t \quad t,x \quad x} \tag{1}$$

where $f(t, x)$ and $g(t, x)$ are differentiable function of x and t, and m and n are non-negative integers.

The CDGSK equation [11-13]:

$$u_t \quad (60u^3 \quad 30uu_{xx} \quad u_{4x})_x \quad 0 \tag{2}$$

is a higher-order generalization of the celebrated KdV equation. It is a very important equation in fluid mechanics which can be expressed:

$$(\mathrm{D}_t \mathrm{D}_x \quad D_x^6)(ff) \quad 2f_{tx}f \quad 2f_t f_x \quad 2f_{xxxxxx}f \quad 12f_{xxxxx}f_x \quad 30f_{xxxx}f_{xx} \quad 20f_{xxx}^2 \tag{3}$$

under the transformation $u = (\ln f)_{xx}$. According to generalized bilinear differential equations in [14], we introduce a new kind of bilinear differential operator:

$$\mathrm{D}_{p,t}^m \mathrm{D}_{p,x}^n (ff) \quad (\partial_t \quad \alpha_p \partial_t \ )^m (\partial_x \quad \alpha_p \partial_x \ )^n [f(t,x)g(t \ ,x \ )]|_{t \quad t,x \quad x} \tag{4}$$

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where $\alpha_p^s \quad \begin{matrix} (\ \ 1)^{r_p(s)}, \ s \quad r_p(s) \bmod p, \ s \quad 6k, \ k \quad N \\ (\ \ 1)^{r_p(s) \ \ 1}, \ s \quad r_p(s) \bmod p, \ s \quad 6k, \ k \quad N \end{matrix}$

According to the previous definition, we have:

$$\alpha_3 \quad 1, \ \alpha_3^2 \quad 1, \ \alpha_3^3 \quad 1, \ \alpha_3^4 \quad 1, \ \alpha_3^5 \quad 1, \ \alpha_3^6 \quad 1 \tag{2}$$

$$\mathrm{D}_{3,x}\mathrm{D}_{3,t}(ff) \quad 2f_{xt}f \quad 2f_t f_x, \quad D_{3,x}^6(ff) \quad 20f_{xxx}^2 \tag{6}$$

$$(\mathrm{D}_{3,t}\mathrm{D}_{3,x} \quad D_{3,x}^6)(ff) \quad 2f_{xt}f \quad 2f_t f_x \quad 20f_{xxx}^2 \quad 0 \tag{7}$$

This equation possesses the same bilinear type as the standard CDGSK one. We take a dependent variable transformation:

$$u \quad (\ln f)_x \tag{8}$$

by a general Bell polynomial theory [15]. Then we obtain a CDGSK-like non-linear differential equation:

$$2u_t \quad 20(u_{xx} \quad 2uu_x)^2 \quad 2u_t \quad 20(u_{xx})^2 \quad 80uu_x u_{xx} \quad 80u^2(u_x)^2 \quad 0 \tag{9}$$

which is linked to the generalized bilinear eq. (7). More precisely, by virtue of the transformation (8), the following equality holds:

$$\frac{(\mathrm{D}_{3,x}\mathrm{D}_{3,t} \quad \mathrm{D}_{3,x}^6)(ff)}{f^2} \quad \frac{2f_{xt}f \quad 2f_t f_x \quad 20f_{xxx}^2}{f^2} \quad 2u_t \quad 20(u_{xx} \quad 2uu_x)^2 \quad 0 \tag{10}$$

Two classes of rational solutions

Solution 1. By symbolic computation with MAPLE, we look for polynomial solutions as the form of:

$$f \quad \sum_{i \ 0}^{7}\sum_{j \ 0}^{7} c_{ij}x^i t^j \tag{11}$$

where the $c_{ij}s$ are constants, and find 26 classes of polynomial solutions to the generalized bilinear eq. (7), and we obtain two classes of rational solutions to CDGSK-like equation:

$$u \quad \frac{f_x}{f} \quad \frac{\frac{3c_{03}}{40}t^2x^2 \quad \frac{c_{02}}{20}tx^2 \quad \frac{c_{01}}{40}x^2}{\frac{c_{03}}{40}t^2x^3 \quad \frac{c_{02}}{60}tx^3 \quad c_{03}t^3 \quad \frac{c_{01}}{120}x^3 \quad c_{02}t^2 \quad c_{01}t \quad c_{00}} \tag{12}$$

$$u \quad \frac{f_x}{f} \quad \frac{\frac{c_{01}}{1440}x^4 \quad 4c_{40}x^3 \ \frac{8640c_{40}}{c_{01}}x^2 \quad \frac{2880^2c_{40}^3}{c_{01}}x \quad \frac{360 \ \ 2880^2c_{40}^4}{c_{01}^2}}{c_{01}t \quad \frac{c_{01}}{7200}x^5 \quad c_{40}x^4 \quad \frac{2880c_{40}}{c_{01}}x^3 \quad \frac{2 \ \ 1440^2c_{40}^3}{c_{01}}x^2 \quad \frac{360 \ \ 2880^2c_{40}^4}{c_{01}^2}x \quad c_{00}} \tag{13}$$

Solution 2. We find another class of polynomial solutions to the generalized bilinear eq. (7). The polynomial solutions are the form of:

$$f_m \quad (c_{2,m}x^2 \quad c_{1,m}x \quad c_{0,m})t^m \tag{14}$$

Obviously:

$$f \quad f_0 \quad f_1 \quad f_2 \quad \cdots \quad f_{m \ 1} \quad f_m \quad \sum_{i \ 0}^{2}\sum_{j \ 0}^{m} c_{ij}x^i t^j \tag{15}$$

We have another class of rational solutions to CDGSK-like equation:

$$u = \frac{f_x}{f} = \frac{\sum_{i=1}^{2}\sum_{j=0}^{m} i c_{ij} x^{i-1} t^{j}}{\sum_{i=0}^{2}\sum_{j=0}^{m} c_{ij} x^{i} t^{j}} \tag{16}$$

when $f = f_0 = c_{20}x^2 + c_{10}x + c_{00}$ and $c_{20} = 0$, $c_{10} = 1$, $c_{00} = c$, we obtain the simplest form of rational solutions to CDGSK-like equation:

$$u = \frac{1}{x + c} \tag{17}$$

when $f = f_1 + f_2$ and $c_{21} = c_{11} = c_{01} = 0$, $c_{20} = 2$, $c_{10} = c_{00} = 0$, we have:

$$u \quad \frac{2xt \quad 4x \quad t}{x^2 t \quad xt \quad 2x \quad t} \tag{18}$$

Conclusions

In this paper, we apply the improved Hirota bilinear differential operator to obtain an CDGSK-like non-linear differential equation. Then, we obtain two classes of rational solutions to the CDGSK-like non-linear differential equation, which are generated from polynomial solutions to the corresponding generalized bilinear equation. These solutions are very basic for the rogue wave and lump solution in ocean. In the further research, we can study rogue wave solution and lump solution through rational solution.

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