VARIABLE SEPARATION FOR TIME FRACTIONAL ADVECTION-DISPERSION EQUATION WITH INITIAL AND BOUNDARY CONDITIONS

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In this paper, variable separation method combined with the properties of Mittag-Leffler function is used to solve a variable-coefficient time fractional advection-dispersion equation with initial and boundary conditions. As a result, a explicit exact solution is obtained. It is shown that the variable separation method can provide a useful mathematical tool for solving the time fractional heat transfer equations.

Key words: *advection-dispersion equation, variable separation method, Mittag-Leffler function, exact solution*

Introduction

It is well known that many phenomena in engineering, physics, chemistry, economics, and other fields can be described very successfully using fractional differential equations. This make fractional calculus play a significant role in describing these phenomena. One of the most important fractional differential equation often used in engineering is the fractional advection-diffusion equation [1]. Recently, solving fractional differential equations has attached much attentions [2-5] and some effective methods have been proposed for fractional advection-diffusion equation, such as finite element method [6], Adomian decomposition method [7], homotopy perturbation method [8], and variational iteration method [9]. However, searching for exact analytical solutions of non-linear fractional differential equations is still on a preliminary stage [10]. When the inhomogeneities of media and non-uniformities of boundaries are taken into account, the variable-coefficient equations could describe more realistic physical phenomena than their constant-coefficient counterparts [11]. Therefore, how to construct exact solutions of fractional differential equations with variable coefficients is worth studying. The present paper is motivated by the desire to extend the variable separation method [12] to the following variable-coefficient time fractional advection-dispersion equation [1]:

$$\frac{\partial^\alpha u(x,t)}{\partial t} \quad v(x,t)u_{xx}(x,t) \quad d(x,t)u_x(x,t) \quad f(x,t), \quad 0 \quad t \quad T, \quad L \quad x \quad R \tag{1}$$

subject to new initial and boundary conditions:

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$$u(x,0)=0,\quad L\le x\le R \tag{2}$$

$$u(L,t)=0,\quad u(x,0)=0,\quad u_t(R,t)=\varphi(t) \tag{3}$$

where $0<\alpha\le 1$ is a parameter describing the fractional derivative in the Caputo sense, $v, d>0$, *i. e.*, liquid is from left to right. The Caputo's fractional derivative is defined [13]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial \tau^m}\mathrm{d}\tau \tag{4}$$

for $m-1<\alpha\le m, m\in N, x>0$. In this paper, some properties of the Caputo time-fractional derivative are used:

$$\mathrm{D}_t^\alpha[\lambda f(t)+\mu g(t)]=\lambda \mathrm{D}_t^\alpha f(t)+\mu \mathrm{D}_t^\alpha g(t) \tag{5}$$

$$\mathrm{D}_t^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha},\quad \gamma>0 \tag{6}$$

$$\mathrm{D}_t^\alpha E_\alpha(qt^\alpha)=qE_\alpha(t^\alpha) \tag{7}$$

where λ, μ, and q are constants or functions independent of t, $E_\alpha(\cdot)$ – the Mittag-Leffler function defined by:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)} \tag{8}$$

Exact solutions

Here, we use the variable separation method [12] to solve eq. (1). To begin with, we take a transformation:

$$u=g(t)w(x),\quad f(x,t)=f(t)w(x) \tag{9}$$

then eq. (1) is converted into:

$$\frac{\partial^\alpha [g(t)w(x)]}{\partial t^\alpha}=-v(x,t)g(t)w'(x)+d(x,t)w''(x)+f(t)w(x) \tag{10}$$

Further supposing that:

$$\frac{\partial^\alpha [g(t)w(x)]}{\partial t^\alpha}-f(t)w(x)=0 \tag{11}$$

then we get a solution of eq. (11):

$$g(t)=\frac{A}{\Gamma(\alpha)}t^{\alpha-1}+\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(1+k+\alpha)}t^{k+\alpha},\quad 0<t\le d\le T \tag{12}$$

Here $f(t)$ is supposed can be expanded in Taylor series:

$$f(t)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}t^k \tag{13}$$

The $g(t)$ has the initial condition $\mathrm{D}_t^{\alpha-1}g(t)|_{t=0}=A$ and converges for $0<t\le d$, $\mathrm{D}_t^{\alpha-1}$ – the Caputo fractional derivative operator, A – a constant, and d – the radius of convergence.

Under the previous assumptions, eq. (10) is reduced to:

$$[-v(x,t)w'(x)+d(x,t)w''(x)]g(t)=0 \tag{14}$$

Solving eq. (14), we have:

$$w(x)=\int_L^x e^{\int_L^x \frac{v(s,t)}{d(s,t)}\mathrm{d}s+c_1}\mathrm{d}x+c_2 \tag{15}$$

and hence obtain an exact solution of eq. (1):

$$u=\frac{A}{\Gamma(\alpha)}t^{\alpha-1}+\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(1+k+\alpha)}t^{k+\alpha}+\int_L^x e^{\int_L^x \frac{v(s,t)}{d(s,t)}\mathrm{d}s+c_1}\mathrm{d}x+c_2 \tag{16}$$

where the constants A, c_1, c_2, and the function $f(t)$ are determined by eqs. (2) and (3).

Example

Here, we further determine solution (16) through the example:

$$\mathrm{D}_t^{\alpha}u=\sinh(x-t)u_{xx}-\sinh(x-t)u_x+e^t\frac{1}{\Gamma(\alpha)}t^{\alpha-1}+\sum_{k=0}^{\infty}\frac{1}{\Gamma(1+k+\alpha)}t^{k+\alpha} \tag{17}$$

subject to the initial and boundary conditions:

$$u(x,0)=0,\quad 0<t<1,\ 0<x<2 \tag{19}$$

$$u(0,t)=0,\quad u(x,0)=0,\quad \mathrm{D}_t^{\alpha}u(2,t)=(t^{-1}+e^t)(e^2-1) \tag{20}$$

In this case, it is easy to see:

$$f(t)=e^t,\quad w(x)=e^x-1,\quad c_1=0,\quad c_2=0 \tag{21}$$

and that eq. (1) has a solution in the form:

$$u=\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1}+\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\alpha+1)}t^{k+\alpha}\right](e^x-1) \tag{22}$$

When we set α = 1, solution (22) becomes:

$$u=e^t(e^x-1) \tag{23}$$

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