# ON FRACTAL SPACE-TIME AND FRACTIONAL CALCULUS

by

***Yue HU*** [a*] ***and Ji-Huan HE*** [b]

[a] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China
[b] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China

Original scientific paper
DOI: 10.2298/TSCI1603773H

*This paper gives an explanation of fractional calculus in fractal space-time. On observable scales, continuum models can be used, however, when the scale tends to a smaller threshold, a fractional model has to be adopted to describe phenomena in micro/nano structure. A time-fractional Fornberg-Whitham equation is used as an example to elucidate the physical meaning of the fractional order, and its solution process is given by the fractional complex transform.*

Key words: *He's fractional derivative, Adomian's decomposition method, variational iteration method, exp-function method, fractional complex transform, fractional Fornberg-Whitham equation*

## Introduction

Is space-time smooth and continuous?

The answer is definitely NO, the value of fractal dimensions of our space-time is neither 4 nor 3 + 1, it is [1-4]:

$$D_4 \quad 4 \quad \phi^3 \quad 4 \quad \frac{1}{4 \quad \frac{1}{4 \quad \frac{1}{4 \quad \ddots}}} \quad \frac{1}{\phi^3} \quad 4.236 \tag{1}$$

where $\phi \quad (\sqrt{5} \quad 1)/2$.

The fractal explanation of eq. (1) is given in fig. 1 [4, 5], which looks like a Russian doll with self-similarity in all scales.

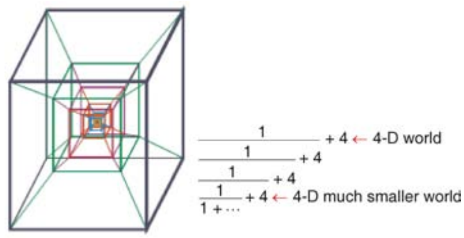


**Figure 1. Fractal space-time model with self-similarity [5]**

To demonstrate the discontinuity of space and time, we consider a TV screen (fig. 2) that is smooth at any ordinary observable scales. However, when the scale becomes smaller and smaller, until to a very small one, the surface reveals an unsmooth face consisting of many arrayed pixels with fractal dimensions larger

* Corresponding author; e-mail: huu3y9@163.com

**Figure 2. The *smooth* TV screen is not smooth at small scales**

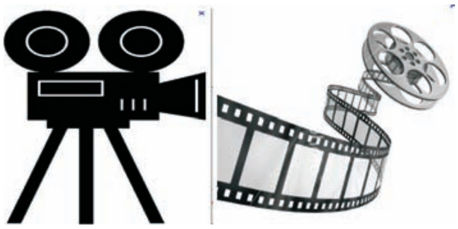

**Figure 3. *Continuous* movement in a film consists of discontinuous slips**

than two. Time is also discontinuous when it is extremely short. A film gives 24 or more slips per second, this gives a continuous movement, fig. 3. However, in the case of 20 or less slips per second, the movement becomes discontinuous.

## Fractal *vs.* continuous space-time

We have been using Newton's law for centuries without any error for any observable scales, but when the scale tends to a smaller and smaller value, Newton's laws will become invalid. Majumder *et al.* [6] found that liquid flow through a membrane composed of an array of aligned carbon nanotubes is 4-5 orders of magnitude faster than would be predicted from conventional fluid-flow theory. The experimental observation reveals that Newton's law for continuum media is not valid for nanoscale flows [7], which always have extremely properties, but this does not mean that mass conversation and other nature laws become invalid, it means that the continuum model becomes invalid. So when the scale tends to an extremely small value, a discontinuous model has to be adopted. In practical applications, all unsmooth surface can be treated as a fractal boundary, and the fractal dimensions can be easily calculated, see examples in [8-10], and fractional calculus [11-19] can describe all physical properties in fractal space-time, and the value of the fractal dimensions is corresponding to the fractional order [20].

Now the problem arises: when should we use the fractional model in practical applications? We give an example for a hierarchic fabric with multiple layers as illustrated in fig. 4. We find the surface is smooth when we touch it, because the scale is too large to feel micro-structure of the fabric. However, if we want to study its air permeability or heat transfer through the fabric, we have to consider the smallest porosity in the cascade of the hierarchic fabric. Assume the smallest scale is, $L_0$, the side length of black square in fig. 4, and any scales smaller than $L_0$ become meaningless in practical applications.

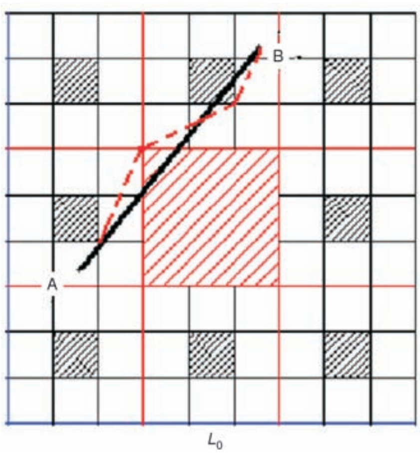


**Figure 4. A hierarchic fabric with multiple layers** (*for color image see journal web-site*)

The fractal derivative [4, 21] can be defined:

$$\frac{Du}{Dx^{\alpha}} = \Gamma(1+\alpha)\lim_{\Delta x = x_A - x_B \to L_0} \frac{u(A)-u(B)}{(x_A - x_B)^{\alpha}} \quad (2)$$

Please note $\Delta x$ tends to $L_0$, not zero as always defined in any a mathematics textbook.

Equation (2) can use to model air permeability in hierarchic porous media [22-25] with great success, it can also optimally determine the value of $L_0$ in a fractal hierarchy [26].

## Fractional complex transform

When the scale $\Delta x \gg L_0$, the surface becomes smooth, and a continuum model can be used. When, $L_0 < \Delta x < 3L_0$, the red square in fig. 4, any fractal derivative or fractional derivative cannot describe any phenomena happened in the cascade with black squares in fig. 4. So scale is everything, different scales results in different fractal dimensions and the order of fractional derivative. To demonstrate this, we consider an A4 format paper. If the scale is the length of the short side of the rectangle, A4 paper is 2-D, while if the scale is the length of long side, the width dimension is disappeared, and the A4 paper becomes 1-D. If the scale becomes as small as the thickness, the A4 paper is 3-D!

In a fractal space, fig. 4, the distance of the discontinuous line of AB can be written in the form scale:

$$\Delta X \quad k(\Delta x)^{\alpha} \tag{3}$$

where $k$ is a scaling parameter, $\Delta x = L_0$, and $\alpha$ – the value of fractal dimensions.

Equation (3) implies that a transform of fractal space to a continuous partner when the scale is $\Delta X$. For simplicity and consistency, we re-write eq. (3) in the form:

$$\Delta X \quad \frac{(\Delta x)^{\alpha}}{\Gamma(\alpha \quad 1)} \tag{4}$$

when $\alpha = 1$, we have $\Delta X = \Delta x$.

Equation (4) is the well-known fractional complex transform [27, 28], the scale, $\Delta X$, adopts a continuum model.

## Fractional calculus

The most important factor in fractional calculus is how to define the fractional derivative. As previously mentioned, fractional derivative is to model phenomena in fractal media, so the definition should be followed its physical insight. Accordingly, [4] gave the following definition:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} \quad \frac{1}{\Gamma(n \quad \alpha)} \frac{d^{n}}{dt^{n}} \int_{t_0}^{t} (s \quad t)^{n \quad \alpha \quad 1} [u_0(s) \quad u(s)] ds \tag{5}$$

Note that $u_0$ is the solution of its continuum partner with same boundary/initial conditions. Applications of eq. (5) can be found in [29, 30].

As an example, we consider a fractional Fornberg-Whitham equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} \quad \frac{\partial^{2 \quad \alpha} u}{\partial x^2 \partial t^{\alpha}} \quad \frac{\partial u}{\partial x} \quad u \frac{\partial u}{\partial x} \quad 3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \quad \frac{\partial^3 u}{\partial x^3} \tag{6}$$

with the following initial condition:

$$u(x,0) \quad \exp \frac{1}{2} x \tag{7}$$

Fornberg-Whitham equation is a non-local model for non-linear dispersive waves. Its fractional partner, eq. (6), describes some a non-linear dispersive wave with discontinuous time. A wave on sea surface will be affected by the motion of the Moon, and $\alpha$ is the fractal dimensions of the Moon's trajectory under the reference system of the Sun. Therefore, eq. (6) can study the effect of the Moon's motion on dispersive waves.

Using the fractional complex transform:

$$T \quad \frac{t^{\alpha}}{\Gamma(\alpha \quad 1)} \tag{8}$$

We can easily convert eq. (6) into a differential equation, which is the following form:

$$\frac{\partial u}{\partial T} \quad \frac{\partial^3 u}{\partial x^2 \partial T} \quad \frac{\partial u}{\partial x} \quad u\frac{\partial u}{\partial x} \quad 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \quad u\frac{\partial^3 u}{\partial x^3} \tag{9}$$

Equation (9) can be solved by the variational iteration method [31-35], the homotopy perturbation method [34, 35], and the exp-function method [34, 35]. In this paper, the Adomian's decomposition method [36] is used, and the following approximate solution is obtained:

$$u(x,T) \quad \exp \quad \frac{1}{2}x \quad 1 \quad \frac{85}{128}T \quad \frac{27}{128}T^2 \quad \frac{7}{192}T^3 \quad \frac{1}{384}T^4 \quad \cdots \tag{10}$$

or

$$u(x,t) \quad \exp \quad \frac{1}{2}x \quad 1 \quad \frac{85}{128} \quad \frac{t^{\alpha}}{\Gamma(\alpha \quad 1)} \quad \frac{27}{128} \quad \frac{t^{\alpha}}{\Gamma(\alpha \quad 1)}^{2}$$

$$\frac{7}{192} \quad \frac{t^{\alpha}}{\Gamma(\alpha \quad 1)}^{3} \quad \frac{1}{384} \quad \frac{t^{\alpha}}{\Gamma(\alpha \quad 1)}^{4} \quad \cdots \tag{11}$$

## Conclusions

There are many definitions of fractional derivatives, we adopt eq. (5) for our study. The fractional complex transform is used to convert the fractal space-time to its continuous partner, and all known analytical methods can be directly applied to the resultant equations. This paper is an explanation of fractional calculus in a fractal frame.

## References

[1] He, J.-H., An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering, *Int. J. Mod. Phys. B, 22* (2008), 21, pp. 3487-3578
[2] El Naschie, M. S., Elementary Prerequisites for E-Infinity (Recommended Background Readings in Nonlinear Dynamics, Geometry and Topology), *Chaos, Solitones & Fractals, 30* (2006), 3, pp. 579-605
[3] El Naschie, M. S., A Review of E-Infinity Theory and the Mass Spectrum of High Energy Particle Physics, Chaos, *Solitones & Fractals, 19* (2004), 1, pp. 209-236
[4] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[5] He, J.-H., Hilbert Cube Model for Fractal Spacetime, *Chaos, Solitones & Fractals, 42* (2009), 5, pp. 2754-2759
[6] Majumder, M., *et al*., Nanoscale Hydrodynamics: Enhanced Flow in Carbon Nanotubes, *Nature, 438* (2005), Nov., p. 44
[7] Chen, R. X., *et al*., Silk Cocoon: "Emperor's New Clothes" for Pupa: Fractal Nano-Hydrodynamical Approach, *Journal of Nano Research, 22* (2013), May, pp. 65-70
[8] Kong, H. Y., *et al*., The Fractal Harmonic Law and its Application to Swimming Suit, *Thermal Science, 16* (2012), 5, pp. 1467-1471
[9] Kong, H. Y., *et al*., A Novel Friction Law, *Thermal Science, 16* (2012), 5, pp. 1529-1533
[10] Fan, J., *et al*., Model of Moisture Diffusion in Fractal Media, *Thermal Science, 19* (2015), 4, pp. 1161-1166
[11] Wu, G. C., *et al*., Lattice Fractional Diffusion Equation in Terms of a Riesz-Caputo Difference, *Physica A, 438* (2015), Nov., pp. 335-339

[12] Wu, G. C., *et al*., Discrete Fractional Diffusion Equation, *Nonlinear Dynamics, 80* (2015), 1, pp. 281-286
[13] Chen, F. L., A Review of Existence and Stability Results for Discrete Fractional Equations, J*ournal of Computational Complexity and Applications, 1* (2015), 1, pp. 22-53
[14] Zhang, M. F., *et al*., Efficient Homotopy Perturbation Method for Fractional Nonlinear Equations Using Sumudu Transform, *Thermal Science, 19* (2015), 4, pp. 1167-1171
[15] Cui, L. X., *et al.*, New Exact Solutions of Fractional Hirota-Satsuma Coupled Korteweg-De Vries Equation, *Thermal Science, 19* (2015), 4, pp. 1173-1176
[16] Qin, Y. M., *et al*., Discrete Fractional Diffusion Model with Two Memory Terms, *Thermal Science, 19* (2015), 4, pp. 1177-1181
[17] Ma, H. C., *et al*., Exact Solutions of Nonlinear Fractional Partial Differential Equations by Fractional Sub-Equation Method, *Thermal Science, 19* (2015), 4, pp. 1239-1244
[18] Wu, F., Liu, J. F., Discrete Fractional Creep Model of Salt Rock, *Journal of Computational Complexity and Applications, 2* (2016), 1, pp. 1-6
[19] Keshtkar, F., On Global Stability of Nonlinear Fractional Dynamical Systems, *Journal of Computational Complexity and Applications, 2* (2016), 1, pp. 16-23
[20] He, J.-H., *et al*., Geometrical Explanation of the Fractional Complex Transform and Derivative Chain Rule for Fractional Calculus, *Physics Letters A, 376* (2012), 4, pp. 257-259
[21] He, J.-H., A New Fractal Derivation, *Thermal Science, 15* (2011), Suppl. 1, pp. S145-S147
[22] Fan, J., He, J.-H., Fractal Derivative Model for Air Permeability in Hierarchic Porous Media, *Abstract and Applied Analysis, 2012* (2012), ID354701, p. 7
[23] Fan, J., He, J.-H., Biomimic Design of Multi-Scale Fabric with Efficient Heat Transfer Property, *Thermal Science, 16* (2012), 5, pp. 1349-1352
[24] Fan, J., Shang, X. M., Water Permeation in the Branching Channel Net of Wool Fiber, *Heat Transfer Research, 44* (2013), 5, pp. 465-472
[25] Fan, J., Shang, X. M., Fractal Heat Transfer in Wool Fiber Hierarchy, *Heat Transfer Research, 44* (2013), 5, pp. 399-407
[26] Fei, D. D., *et al.*, Fractal Approach to Heat Transfer in Silkworm Cocoon Hierarchy, *Thermal Science, 17* (2013), 5, pp. 1546-1548
[27] He, J.-H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science, 16* (2012), 2, pp. 331-334
[28] Li, Z., He, J.-H., Fractional Complex Transform for Fractional Differential Equations, *Math. Comput. Appl, 15* (2010), 5, pp. 970-973
[29] He, J.-H., *et al*., A New Fractional Derivative and its Application to Explanation of Polar Bear Hairs, *Journal of King Saud University Science* (on line first), 2015
[30] Liu, F. J., *et al.*, He's Fractional Derivative for Heat Conduction in a Fractal Medium Arising in Silkworm Cocoon Hierarchy, *Thermal Science, 19* (2015), 4, pp. 1155-1159
[31] Liu, J. F., Modified Variational Iteration Method for Varian Boussinesq Equation, *Thermal Science, 19* (2015), 4, 1195-1199
[32] Huang, L. L., Matrix Lagrange Multiplier of the VIM, *Journal of Computational Complexity and Applications, 2* (2016), 3, pp. 86-88
[33] Wu, G. C., New Correction Functional and Lagrange Multipliers for Two Point Value Problems, *Journal of Computational Complexity and Applications, 2* (2016), 2, pp. 46-48
[34] He, J.-H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis, 2012* (2012), ID916793
[35] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B, 20* (2006), 10, pp. 1141-1199
[36] Adomian, G., A Review of the Decomposition Method in Applied Mathematics, *J. Math. Anal. Appl, 135* (1988), 2, pp. 44-501

Paper submitted: September 10, 2015
Paper revised: December 12, 2015
Paper accepted: December 12, 2015