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TRANSIENT HEAT DIFFUSION WITH A NON-SINGULAR FADING MEMORY
From the Cattaneo Constitutive Equation with Jeffrey's Kernel to the Caputo-Fabrizio Time-Fractional Derivative

by

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Starting from the Cattaneo constitutive relation with a Jeffrey's kernel the derivation of a transient heat diffusion equation with relaxation term expressed through the Caputo-Fabrizio time fractional derivative has been developed. This approach allows seeing the physical background of the newly defined Caputo-Fabrizio time fractional derivative and demonstrates how other constitutive equations could be modified with non-singular fading memories.

Key words: *non-linear diffusion, non-singular fading memory, Jeffrey kernel, Caputo-Fabrizio derivative integral balance approach*

Introduction

This article refers to a hot topic in modelling of dissipative phenomena. As it is stated in the seminal work of Caputo and Fabrizio [1] many classical constitutive equations (see the comments in [1] and the references therein) are not always adequate to behaviour of all new materials appearing in the modern technology. To satisfy these requirements a new time-fractional derivative with a non-singular smooth exponential and kernel was conceived [1]:

$$_{CF}\mathrm{D}_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \frac{\mathrm{d}f(t)}{\mathrm{d}t}\,\mathrm{d}s =$$

$$= \frac{1}{1-\alpha} \int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \frac{\mathrm{d}f(t)}{\mathrm{d}t}\,\mathrm{d}s \qquad (1\mathrm{a, b})$$

where $M(\alpha)$ in eq. 1(a) is a normalization function such that $M(0) = M(1) = 1$. In accordance with this definition if $f(t) = C = $ const., then $_{CF}\mathrm{D}_t^\alpha f(t) = 0$ as in the classical Caputo derivative [2], but now the kernel has no singularity. Losada and Nieto [3] demonstrated that for special case the normalization function is function $M(\alpha) = 2/(2 - \alpha)$. In the seminal article of

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Caputo and Fabrizio it was assumed for convenience that $M(\alpha) = 1$ that leads to the final definition of the new time-fractional derivative [1, 3, 4] in the form of eq. 1(b).

The Caputo-Fabrizio time-fractional derivative immediately attracted the interest of the researchers and for about a year after the seminal publication [1] numerous applications have been reported, among them: elasticity [4], resistance and numerical modelling of fractional electric circuit [5, 6], the Keller-Segel model [7], Fisher's reaction-diffusion equation [8], coupled systems of time-fractional differential problems [9], mass-spring damped systems [10], groundwater flow [11], *etc.*

This note refers to the Cattaneo concept of flux relaxation with a Jeffrey's exponential kernel in view of its relationship to heat diffusion in terms of the Caputo-Fabrizio time-fractional derivative.

The Cattaneo heat diffusion with Jeffrey's kernel

Diffusion phenomena, of heat or mass, are generally described as a consequence of the conservation law by the relationship

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}, \qquad q(x, t) = -k \frac{\partial T(x, t)}{\partial x} \Rightarrow \rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \qquad \text{(2a,b,c)}$$

with the assumption (2b) that the flux $q(x, t)$ is proportional to the gradient we get the definition of the heat conductivity, k. From eqs. (2a) and (2b) we get the Fourier (Fick) law (2c), but this equation defines an infinite speed of propagation of the flux which is unphysical.

A damping function related to a finite speed of heat diffusion in rigid conductors was conceived by Cattaneo [12] by a generalization of the Fourier law through a linear superposition of the heat flux and its time derivative related to its history [13, 14]. With this concept, the heat flux, q, satisfy the constitutive equation [12] incorporating its past history related to the time-delay, s:

$$q(x, t) = -\int_{-\infty}^{t} R(x, t) \nabla T(x, t - s) \mathrm{d}s \qquad (3)$$

For homogeneous rigid heat conductors, $R(x, t)$ is space-independent and can be represented by the Jeffrey kernel [1, 15] $R(t) = \exp[-(t - s)/\tau]$ where the relaxation time, τ, is finite, *i. e.* τ = constant. Then, the energy balance yields the Cattaneo equations [12]:

$$\frac{\partial T(x, t)}{\partial t} = -\frac{a_2}{\tau} \int_{0}^{t} \exp\left[-\left(\frac{t - s}{\tau}\right)\right] \frac{\partial T(x, s)}{\partial x} \mathrm{d}s, \qquad a_2 = \frac{k_2}{\rho C_p} \qquad (4)$$

For $\tau \rightarrow 0$ the limit of the Cattaneo equation is the Fourier law. Therefore, in the first order approximation, in τ, the modified Fourier law is [16]:

$$q(x, t + \tau) = -k_1 \frac{\partial T(x, t)}{\partial x}, \qquad q(x, t + \tau) \approx q(x, t) + \tau \frac{\partial q(x, t)}{\partial x} \qquad \text{(5a,b)}$$

This leads to a first order differential equation [16]:

$$\frac{1}{\tau} q(x, t) + \frac{\partial q(x, t)}{\partial t} = -\frac{k_1}{\tau} \frac{\partial T(x, t)}{\partial x} \qquad (6)$$

The integration of eq. (6) results in the Cattaneo eq. (3), which is the simplest giving rise to finite speed of flux propagation. Joseph and Preziosi [15] have considered a modified

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relaxation function replacing the exponential kernel in (3) by $R_{JP} = k_1\delta(s) + k_2/\tau \exp(-s/\tau)$, where $\delta(s)$ is Dirac delta function , while k_1 and k_2 are the *effective thermal conductivity* and the *elastic conductivity*, respectively. In this case the Fourier law leads to a flux defined [15, 16]:

$$q(x,t) = -k_1 \frac{\partial T(x,t)}{\partial x} - \frac{k_2}{\tau} \int_{-\infty}^{t} e^{-\left(\frac{t-s}{\tau}\right)} \frac{\partial T(x,t)}{\partial x} ds \qquad (7)$$

In this case, the energy conservation equation of the internal energy [17] results in the Jeffrey type integro-differential equation [16]:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{a_2}{\tau} \int_{-\infty}^{t} e^{-\left(\frac{t-s}{\tau}\right)} \frac{\partial^2 T(x,s)}{\partial x^2} ds, \qquad a_1 = \frac{k_1}{\rho C_p}, \qquad a_2 = \frac{k_2}{\rho C_p} \quad (8a)$$

or equivalently:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2\beta \int_{-\infty}^{t} e^{-\beta(t-s)} \frac{\partial^2 T(x,t)}{\partial x^2} ds \qquad (8b)$$

**A step towards to a time-fractional derivative
with exponential kernel**

For the sake of simplicity, let us consider a virgin material subjected to a thermal load at $x = 0$, that is the following initial and boundary conditions take place:

$$T(x, 0) = T(0.0) = T(\infty, t) = T_x(x, 0) = T_{xx}(x, 0) = 0, \qquad T(0, t) = T_s \qquad (9)$$

Denoting, $F(x, t) = \partial^2 T(x,t)/\partial x^2$ and after integration by parts of the last term of eq. (8b) we get (omitting the cumbersome expressions):

$$\beta \int_{-\infty}^{t} e^{-\beta(t-s)} F(x,s) ds = e^{-\beta(t-s)}[F(x,s) - F(x,t)]\Big|_{-\infty}^{t} + \beta \int_{-\infty}^{t} e^{-\beta(t-s)}[F(x,t) - F(x,s)] ds \quad (10)$$

The first term in the RHS of eq. (10) is zero, while the second one matches the definition of the Caputo-Fabrizio fractional derivative [3], namely:

$$_{CF}D_t^\alpha = \frac{\alpha}{(1-\alpha)^2} \int_{-\infty}^{t} [f(t) - f_a(s)] \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right] ds, \quad t > 0 \qquad (11)$$

At the moment we term it a *pro-Caputo* (non-normalized) derivative denoted as $_{PC}D_t^\beta$, expressed in two equivalent forms (in accordance with the notations used in ref. [1]), namely:

$$_{PC}D_t^\beta F(x,t) = \beta \int_{-\infty}^{t} e^{-\beta(t-s)}[F(x,t) - F(x,s)] ds = \beta \int_{-\infty}^{t} e^{-\beta(t-s)} \frac{dF(x,s)}{dt} ds \qquad (12a,b)$$

Obviously, the rate constant β in (12a,b) controls the kernel and $\beta \in (0, \infty)$. If we like to refine $_{PC}D_t^\beta$ as an integral operator controlled by a single parameter α in a subdiffusive

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manner, we have to satisfy the conditions: for $\alpha \in [0.1] \Rightarrow 1/\beta \in [0,\infty]$. With $\beta(\alpha) =$ $= \alpha/(1-\alpha)$ [1, 2] the desired properties are obtained, namely:

$$\frac{1}{\beta} = \frac{1-\alpha}{\alpha} \in [0,\infty], \quad \alpha = \frac{1}{1+\dfrac{1}{\beta}} = \frac{\beta}{1+\beta} \in [0,1], \quad \frac{\alpha}{(1-\alpha)^2} = \frac{\beta}{1-\alpha} \qquad (13a,b,c)$$

Further, following the definition of the Caputo-Fabrizio derivative [1, 2] and considering the lower limit of integral at $t = 0$, *i. e. a* = 0, we have:

$$_{CF}D_t^\alpha T(x,t) = \frac{N(\sigma)}{\sigma}\,_{PC}D_t^\beta T(x,t) = \frac{N(\sigma)}{\sigma}\int_0^t e^{\frac{-\alpha}{1-\alpha}(t-s)}\frac{dF(x,s)}{dt}ds =$$

$$= \frac{M(\alpha)}{1-\alpha}\int_0^t e^{\frac{-\alpha}{1-\alpha}(t-s)}\frac{dF(x,s)}{dt} \qquad (14a,b,c)$$

In the terms used here $\sigma = 1/\beta$, while $N(\sigma)$ and $M(\alpha)$ are normalization functions [1, 2]. With $M(\alpha) = 1$, as mentioned at the beginning, the form (14c) reduces to Caputo-Fabrizio time-fractional derivative of $F(x, t)$ defined by eq. (1b). Now, turning on eq. (8b) in terms of $T(x, t)$ an taking into account (11) and (13c) the last term is the Caputo-Fabrizio time-fractional derivative of $\partial^2 T(x,t)/\partial x^2$, namely:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,_{CF}D_t^\alpha \frac{\partial^2 T(x,t)}{\partial x^2}, \quad t > 0 \qquad (15)$$

Equation (15) models transient heat conduction with a damping term expressed through the Caputo-Fabrizio fractional derivative. Clearly, for $\alpha = 1$ we get the Fourier equation and decreasing α that physically means an increase of the damping effect the weight of the last term increases, too. If the Jeffrey kernel is only taken into account (the modified relaxation function $R_{JP}(t)$ of Joseph and Preziosi [15] is omitted), then the equivalent form of eq. (15) is eq. (4) [16], but it *accounts only the elastic heat diffusion* and can be expressed in terms of $_{PC}D_t^\alpha$:

$$\frac{\partial T(x,t)}{\partial t} = -a_2\beta\int_0^t \exp[-\beta(t-s)]\frac{\partial T(x,s)}{\partial x}ds \Rightarrow \frac{\partial T(x,t)}{\partial t} = -a_2\frac{\alpha}{1-\alpha}\,_{PC}D_t^\alpha\left[\frac{\partial T(x,s)}{\partial x}\right] (16a,b)$$

The finite penetration depth and the front propagation:
Integral-balance approach

The Cattaneo concept of the relaxations function $R(x, t)$ means a finite rate of the flux penetrating the medium which immediately leads to the concept of a finite penetration depth of the thermal field. Let us consider the transient heat conduction equation in a semi-infinite medium represented by eq. (8) with initial and boundary conditions:

$$T(x, 0) = 0, \quad 0 \le x \le \infty, \quad T(0, t) = T_s(t), \quad T(\infty, t) = 0 \qquad (17a,b,c)$$

The finite penetration depth δ_J define a sharp front with conditions replacing $T(\infty, t) = 0$, namely:

$$T(\delta_J, t) = T(0, t) \quad \text{and} \quad \frac{\partial T}{\partial x}(\delta_J, t) = 0 \tag{18a,b}$$

The solution of this problem has been developed in [18] by the heat-balance integral method and assumed profile parabolic profile $T_a(x, t) = T_s(1 - x/\delta)^n$ [19]. Then, the front propagates:

$$\delta_J = \sqrt{2n(n+1)}\sqrt{a_1 t}\sqrt{1 + \frac{a_2}{a_1}\frac{\tau}{t}e^{-\frac{t}{\tau}}} \Rightarrow \delta_J \equiv \sqrt{a_1 t}\, f_R(t, \tau) \tag{19a,b}$$

Therefore, with $k_2 = a_2 = 0$ or when $\tau/t \to 0$, large times or extremely short relation times $\tau \to 0$ which gives that the relaxation term vanishes, $i.\ e.$ for $f_R(t, \tau) \to 0$, we get the classical diffusion equation and the long-time result [6, 7] $\delta_0 \equiv (a_1 t)^{1/2}$.

Now, we refer to the double integration method [20]. In terms of Caputo-Fabrizio derivative the double integral-balance relationship is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\delta_J}\left[\int_x^{\delta_J} T(x,t)\mathrm{d}x\right]\mathrm{d}x = \int_0^{\delta_J} a_1\left[\int_x^{\delta_J}\frac{\partial^2 T(x,t)}{\partial x^2}\mathrm{d}x\right]\mathrm{d}x + a_2(1-\alpha)\int_0^{\delta_J}\left[\int_x^{\delta_J}{}_{CF}\mathrm{D}_t^\alpha\frac{\partial^2 T(x,t)}{\partial x^2}\mathrm{d}x\right]\mathrm{d}x \tag{20}$$

Consequently, with $T_a(x, t) = T_s(1 - x/\delta)^n$ the equation about the penetration depth is:

$$\frac{1}{(n+1)(n+2)}\frac{\mathrm{d}\delta_J^2}{\mathrm{d}t} = a_1 T(0,t) + a_2(1-\alpha)\,{}_{CF}\mathrm{D}_t^\alpha T(0,t) \tag{21}$$

Equation (21) is simple, but if $T(0, t)$ is constant (Dirichlet problem) then the last term is zero. Therefore, the integral-balance approach cannot handle correctly the transient heat conduction expressed through Caputo-Fabrizio time-fractional derivative in the case of the Dirichlet problem. Certainly, if the boundary condition is a function of time $T(0, t) = f(t)$, then the integral-balance method can be applied and the last term in eq. (21) will not be zero, but this problem is beyond the scope of the present research note.

Conclusion

The Cattaneo constitutive equation with Jeffrey's fading memory naturally results in a heat conduction equation with a relaxation term expressed by the Caputo-Fabrizio time-fractional derivative. The example developed allows seeing the physical background of the newly defined Caputo-Fabrizio derivative with non-singular kernel in the case of heat diffusion.

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