THE NON-DIFFERENTIABLE SOLUTION FOR LOCAL FRACTIONAL LAPLACE EQUATION IN STEADY HEAT-CONDUCTION PROBLEM

by

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In this paper, we investigate the local fractional Laplace equation in the steady heat-conduction problem. The solutions involving the non-differentiable graph are obtained by using the characteristic equation method via local fractional derivative. The obtained results are given to present the accuracy of the technology to solve the steady heat-conduction in fractal media.

Key words: *heat-conduction problem, Laplace equation, analytical solution, local fractional derivative*

Introduction

Heat transfer was generalized to the problems in fractal media, *e. g.*, fractal heat transfers [1], and fractal heat conduction [2-7]. The local fractional Laplace equation was used to describe the fractal electrostatics [8-10] and the steady heat-conduction in fractal media [11]. Many methods for solving the local fractional Laplace equation were reported, such as the series expansion method [9], Adomian decomposition method [10, 12], function decomposition method [12], variational iteration method [13], and so on.

In the 2-D case, the local fractional Laplace equation in the steady heat-conduction problem is given by the expression [11]:

$$\frac{\partial^{2\mu}\Phi_\mu(\theta,\vartheta)}{\partial\theta^{2\mu}}+\frac{\partial^{2\mu}\Phi_\mu(\theta,\vartheta)}{\partial\vartheta^{2\mu}}=0 \tag{1}$$

where $\partial^\mu/\partial\theta^\mu$ represent the local fractional partial derivative, which is defined through [7]:

$$\Phi^{(\mu)}_{\mu,\vartheta}(\theta,\vartheta_0)=\frac{\partial^\mu\Phi_\mu(\theta,\vartheta)}{\partial\vartheta^\mu}\Big|_{\vartheta=\vartheta_0}=\lim_{\vartheta\to\vartheta_0}\frac{\Delta^\mu[\Phi_\mu(\theta,\vartheta)-\Phi_\mu(\theta,\vartheta_0)]}{(\vartheta-\vartheta_0)^\mu} \tag{2}$$

with

$$\Delta^\mu[\Phi_\mu(\theta,\vartheta)-\Phi_\mu(\theta,\vartheta_0)]\cong\Gamma(1+\mu)\Delta[\Phi_\mu(\theta,\vartheta)-\Phi_\mu(\theta,\vartheta_0)] \tag{3}$$

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The local fractional derivative of the function $\Phi_\mu(\vartheta)$ of order $\mu\,(0<\mu\le 1)$ at $\vartheta=\vartheta_0$ is defined [7]:

$$D_\vartheta^{(\mu)}\Phi_\mu(\vartheta_0)=\Phi_\mu^{(\mu)}(\vartheta_0)=\frac{\mathrm{d}^\mu\Phi_\mu(\vartheta)}{\mathrm{d}\vartheta^\mu}\Big|_{\vartheta=\vartheta_0}=\lim_{\vartheta\to\vartheta_0}\frac{\Delta^\mu[\Phi_\mu(\vartheta)-\Phi_\mu(\vartheta_0)]}{(\vartheta-\vartheta_0)^\mu} \tag{4}$$

where $\Delta^\mu(\Phi_\mu(\vartheta)-\Phi_\mu(\vartheta_0))\cong\Gamma(1+\mu)\Delta(\Phi_\mu(\vartheta)-\Phi_\mu(\vartheta_0))$.

The local fractional derivative of the Mittag-Leffler function defined on Cantor sets given by:

$$E_\mu(\vartheta^\mu)=\sum_{i=0}^{\infty}\vartheta^{i\mu}/\Gamma(1+i\mu)$$

is [5, 7]:

$$D_\vartheta^{(\mu)}E_\mu(\vartheta^\mu)=E_\mu(\vartheta^\mu) \tag{5}$$

The local fractional Laplace equation in the 3-D fractal space which was described as the steady heat-conduction problem, was written in the form [11]:

$$\nabla^{(2\mu)}\Phi_\mu(\theta,\vartheta,\varphi)=0 \tag{6}$$

where $\nabla^{(2\mu)}$ represents the local fractional Laplace operator [2, 8, 11] and $\Phi_\mu(\theta,\vartheta,\varphi)$ is the temperature in the fractal filed.

The characteristic equation method (CEM) for solving the local fractional differential equations was developed in [14]. In this article, the main aim is to present the application of the CEM to solve the local fractional Laplace equations in the steady heat-conduction problem.

Solving the local fractional Laplace equations in the steady heat-conduction problem

Following the idea of the CEM [14], we now consider the local fractional Laplace equation (1).

We set a proposed Mittag-Leffler solution:

$$\Phi_\mu(\theta,\vartheta)=E_\mu(\rho\theta^\mu)E_\mu(\sigma\vartheta^\mu) \tag{7}$$

which leads to the characteristic equation:

$$\rho^2+\sigma^2=0 \tag{8}$$

From eq. (8) we have:

$$\rho_1=i^\mu|\sigma| \tag{9a}$$

and

$$\rho_2=-i^\mu|\sigma| \tag{9b}$$

Thus, we obtain the general solution of eq. (1) in the Mittag-Leffler function form:

$$\Phi_\mu(\theta,\vartheta) = E_\mu(\sigma\vartheta^\mu)[v_1 E_\mu(i^\mu \,|\, \sigma \,|\, \theta^\mu) + v_2 E_\mu(-i^\mu \,|\, \sigma \tag{10}$$

where v_1 and v_2 are constants.

When $v_1 = v_2 = \sigma = 1$, the Mittag-Leffler solution of non-differentiability defined on Cantor sets is shown in fig. 1.

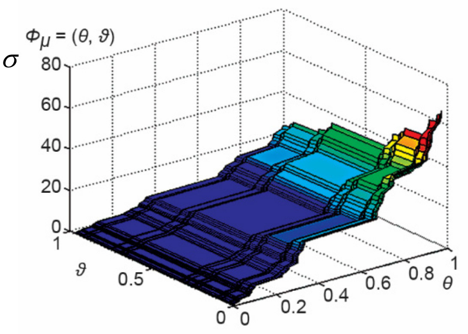

Figure 1. The Mittag-Leffler solution of non-differentiability when $v_1 = v_2 = \sigma = 1$ *(for color image see journal web-site)*

Let us rewrite eq. (6) in the form:

$$\frac{\partial^{2\mu}\Phi_\mu(\theta,\vartheta,\varphi)}{\partial\theta^{2\mu}} + \frac{\partial^{2\mu}\Phi_\mu(\theta,\vartheta,\varphi)}{\partial\vartheta^{2\mu}} + \frac{\partial^{2\mu}\Phi_\mu(\theta,\vartheta,\varphi)}{\partial\varphi^{2\mu}} = 0 \tag{11}$$

where

$$\nabla^{(2\mu)} = \partial^{2\mu}/\partial\theta^{2\mu} + \partial^{2\mu}/\partial\vartheta^{2\mu} + \partial^{2\mu}/\partial\varphi^{2\mu}$$

In a similar way, we suggest a proposed Mittag-Leffler solution in the form:

$$\Phi_\mu(\theta,\vartheta,\varphi) = E_\mu(\rho\theta^\mu)E_\mu(\sigma\vartheta^\mu)E_\mu(\varpi\varphi^\mu) \tag{12}$$

By submitting eq. (12) into eq. (11), we have the characteristic equation:

$$\rho^2 + \sigma^2 + \varpi^2 = 0 \tag{13}$$

Therefore, with the help of eq. (13), we obtain:

$$\rho_1 = i^\mu\sqrt{\sigma^2 + \varpi^2} \tag{14a}$$

and

$$\rho_2 = -i^\mu\sqrt{\sigma^2 + \varpi^2} \tag{14b}$$

From eqs. (15) and (16) we give the general solution of eq. (12) in the Mittag-Leffler function form:

$$\Phi_\mu(\theta,\vartheta,\varphi) = \left[v_1 E_\mu\left(i^\mu\sqrt{\sigma^2+\varpi^2}\,\theta^\mu\right) + v_2 E_\mu\left(-i^\mu\sqrt{\sigma^2+\varpi^2}\,\theta^\mu\right)\right]E_\mu(\sigma\vartheta^\mu)E_\mu(\varpi\varphi^\mu) \tag{15}$$

where v_1 and v_2 are constants.

Conclusion

The local fractional Laplace equations in the steady heat-conduction problem were considered by using the CEM. The non-differentiable solutions for 2-D and 3-D Laplace equations were presented. The results were given to adequately explain the fractal characteristics of the steady heat-conduction in fractal media.

Nomenclature

θ, ϑ – space co-ordinates, [m]

μ – fractal dimension, [–]

$\Phi_\mu(\theta, \vartheta)$ – temperature, [K]

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