## From the Guest Editors

# ANALYTICAL AND NUMERICAL METHODS FOR THERMAL SCIENCE

by

***Qiu-Na CUI*** [a], ***Fu-Juan LIU*** [b], ***and Ji-Huan HE*** [b*]

[a] College of Energy and Architectural Environment Engineering, Henan University of Urban Construction, Pingdingshan, China
[b] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China

*This paper gives a literature review on various analytical methods and numerical methods for heat problems. Fractal models and fractional models are emphasized. Beginning at the classic heat equation, fractional Fourier law and fractional conservation of energy are considered for 1-D heat equation in fractal media, its solution properties are discussed using the fractional complex transform. The emphasis of this literature review is put upon recent publications in Thermal Science, and the references are not exhaustive.*

Key words: *analytical methods, fractal geometry, fractional calculus, fractional complex transform, Leibniz's derivative*

## Introduction

Thermal science is the combined study of thermodynamics, fluid mechanics, air dynamics, heat transfer, surface science, combustion, nanotechnology, environmental science, computer science and mathematics, and it becomes an important role in modern science and technology, for example, the bubble electrospinning [1] and Bubbfil spinning technology [2], which have been used for mass-production of nanofibers, were developed from the thermodynamics of polymer bubbles. Thermal science is also very important for our everyday life, for example, clothing comfort [3] and house heating [4] are applications of thermal science. This issue focuses mainly on analytical methods and numerical methods for practical applications of thermal science. Mathematical models including fractal and fractional models are also emphasized, and the most frontier of nanotechnology is elucidated.

The general heat equation can be written in the form:

$$C\rho \frac{\partial u}{\partial t} \quad \frac{\partial}{\partial x} \; k_1 \frac{\partial u}{\partial x} \quad \frac{\partial}{\partial y} \; k_2 \frac{\partial u}{\partial y} \quad \frac{\partial}{\partial z} \; k_3 \frac{\partial u}{\partial z} \quad Q \tag{1}$$

where $u$ is the temperature, $c$ – the specific heat, $\rho$ – the mass density, $Q$ – the heat source, and $k_i (i = 1, 2, 3)$ – the thermal conductivity.

Equation (1) is valid only for continuous media. For heat problems in discontinuous or fractal media, a fractal model [5-12] or a fractional model [13-18] has to be adopted. 1-D heat equation in a fractal medium can be written in the form [13]:

* Corresponding author; e-mail: hejihuan@suda.edu.cn

$$C\rho \frac{\partial u}{\partial t} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(k \frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right) \tag{2}$$

where $\alpha$ is the fractional dimensions of the porous (fractal) medium, and $\partial^{\alpha}/\partial x^{\alpha}$ – the fractional derivative with order of $\alpha$ defined as [13]:

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} \int_{x_0}^{x} (s-x)^{n-\alpha-1}[u_0(s)-u(s)]\,\mathrm{d}s \tag{3}$$

Note that $u_0$ is the solution of its continuum partner with same boundary/initial conditions. Equation (2) is obtained from the fractional Fourier law:

$$q = -k \frac{\partial^{\alpha} u}{\partial x^{\alpha}} \tag{4}$$

and conservation of energy:

$$C\rho \frac{\partial u}{\partial t} = -\frac{\partial^{\alpha}}{\partial x^{\alpha}}(q) \tag{5}$$

where $q$ is the conduction heat flux.

## Analytical methods

As early as 1822 Fourier studied the following 1-D heat equation [19]:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^{2} u}{\partial x^{2}} \tag{6}$$

where $\beta$ is the thermal diffusivity.

Fourier developed an analytical method for eq. (6), which is now called as the Fourier transform [19]. The method of variation of parameters was originally developed also from eq. (6), and a variational principle was established for eq. (6) in 2009 [20], the He-Lee variational principle for the heat equation given in ref. [20] was caught a hot discussion, which were published in Open Forum of the journal *Thermal Science* from 2013 to 2016 [21-25]. Now there are many analytical methods developed originally for various non-linear heat problems, for examples, the variational iteration method [26, 27], the homotopy perturbation method [28, 29], Adomian method [30] and others. Some useful review articles on analytical methods are available in refs. [31-35].

Analytical methods for fractional heat problems depend mainly on the definitions of fractional derivative. The main difficult is the chain rule, which is not valid for all definitions of fractional derivatives [36, 37]. Generally solutions of fractional differential equations are non-differential anywhere, but in practical applications, measured temperature distribution in porous medium might be smooth enough for some a given scale. This is because any scales smaller than the given scale might be meaningless, and the porous medium is only an approximate fractal geometry. To illustrate this interesting phenomenon, we consider a Koch curve as illustrated in fig.1. When the scale is larger than $L$ (first line of fig.1), the solution is smooth, but it can not describe any phenomena within the scale. For example, a continuum method can not elucidate any properties of fabrics structure. When the scale becomes $L/3$, the solution becomes discontinuous, but the solution within the $L/3$ scale is smooth, *e. g.*, $u(x)$ is smooth when $0 < x < L/3$. Generally, we have the following relationship:

$$X = kx^{\alpha} \tag{7}$$

where $x$ is the scale for study, *i. e.*, $x = L/3$, $x = L/9$, $X$ – the total length of the Koch curve, $k$ – a constant, and $\alpha$ – the fractal dimensions of the curve. In practical applications, the scale can

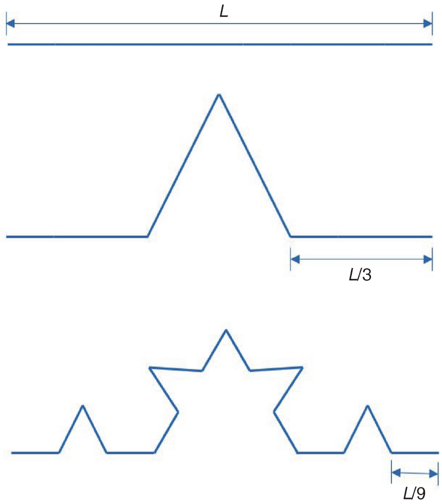


**Figure 1. Koch curve with different scales; the larger scales can not describe any properties appeared in smaller scales; when we use the scale of *L*/3 (the middle), the solution is smooth when 0 < *x* < *L*/3; however, when the scale is *L*/9 (the bottom), the solution becomes discontinuous when 0 < *x* < *L*/3**

never tend to zero, it is a definite value. In the scale of *X*, all discontinuous properties appeared in the scale of *x* disappear. The fractional complex transform [38, 39], which was first proposed in 2010, is to convert a fractal space under the scale of *x* to a smooth space under scale of *X*, so that all analytical methods developed from the advanced calculus can be applied to fractional calculus.

Consider a fractional differential equation in the form:

$$\frac{\mathrm{d}^{\alpha} u(x)}{\mathrm{d}x^{\alpha}} = 0, \quad 0 < \alpha \leq 1 \tag{8}$$

By the fractional complex transform [38, 39]:

$$X = \frac{(px)^{\alpha}}{\Gamma(1+\alpha)} \tag{9}$$

where $p$ is a constant.

Equation (8) turns out to be the following ordinary differential equation:

$$p^{\alpha} \frac{\mathrm{d}u(X)}{\mathrm{d}X} = 0 \tag{10}$$

The solution of eq. (10) is smooth for any scales larger than $\tilde{x} = p\tilde{x}^{\alpha}$, where $\tilde{x}$ is the smallest porous size. Any discontinuous properties for scales smaller than $\tilde{x}$ are ignored.

There are various analytical methods for fractional calculus, among which the variational iteration method, the homotopy perturbation method, the fractional complex transform, Yang-Laplace transform, Yang-Fourier transform, and Hristov's integral-balance method, have received much attention [40-44]. A special issue on fractional calculus was published in the journal *Thermal Science* (volume 19, Supplement 1, 2015) on the Occasion of 60th Anniversary of Professor Jordan Yankov Hristov dedicated to Non-Linear Diffusion Models in Heat and Mass Transfer. A review article on analytical methods for fractional calculus is available in ref. [13].

## Numerical methods

Numerical methods are due to the fast development of computer science, and become a main mathematical tool for analysis of various heat problems. For continuous media, the derivatives are approximated expressed by:

$$\frac{\partial u}{\partial x} \quad \frac{u(x_1) \quad u(x_2)}{x_1 \quad x_2} \tag{11}$$

Equation (11) is similar to Leibniz's derivative. The derivative of $u(x)$ with respect to $x$, in the sense of Leibniz's notation, is the standard part of the infinitesimal ratio:

$$\frac{\partial u}{\partial x} \quad st\,\frac{\Delta u}{\Delta x} \quad st\,\frac{u(x_1) \quad u(x_2)}{x_1 \quad x_2} \tag{12}$$

Leibniz's definition is very close to the definition of the fractal derivative [13]. In a fractal medium, the distance between $x_1$ and $x_2$ tends to infinity ($\Delta x \quad \infty$) even when $x_1 \quad x_2$, and $u$ can be continuous and non-differentiable, therefore Leibniz's work was nearer to the basic properties of modern fractional calculus.

The fractal derivative [5, 13] can be defined as

$$\frac{\mathrm{D}u}{\mathrm{D}x^{\alpha}} \quad \Gamma(1 \quad \alpha) \lim_{\Delta x \; x_1 \; x_2 \quad L_0} \frac{u(x_1) \quad u(x_2)}{(x_1 \quad x_2)^{\alpha}} \quad \Gamma(1 \quad x) \frac{u(x_1) \quad u(x_2)}{kL_0^{\alpha}} \tag{13}$$

Please note $\Delta x$ in eq. (13) tends to $L_0$, not zero as always defined in any mathematics textbook. The fractional models and the fractal derivative models are very close to difference models. Numerical methods based on eq. (13) for discontinuous media are rare and very primary.

## Conclusions

This issue consists mainly of a collection of papers for analytical methods and numerical methods for heat problems, conveying a strong, reliable, efficient, and promising development of thermal science and its development. We hope that this issue will prove to be a timely and valuable reference for researchers in fields of thermal science, nanotechnology, mathematics, and textile engineering as well. In this issue, various advanced analytical methods and numerical methods for real-life heat problems are given and can be used as paradigms for many other applications. The aim of this issue is to bring to the fore the many new and exciting applications of thermal science to the cute frontier of modern technology, thereby capturing both the interest and imagination of the wider communities in various fields.

## Acknowledgments

The work is supported by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), National Natural Science Foundation of China under Grant No. 51463021, and China Postdoctoral Science Foundation under Grant No. 2016T90495.

## References

[1] Chen, R. X., *et al.*, Bubble Rupture in Bubble Electrospinning, *Thermal Science, 19* (2015), 4, pp. 1141-1149

[2] He, C. H., *et al.*, Bubbfil Spinning for Fabrication of PVA Nanofibers, *Thermal Science, 19* (2015), 2, pp. 743-746

[3] Gao, S. W., *et al.*, Near-Infraed Scattering Method for Fabric Thermal Comfort, *Thermal Science, 18* (2014), 5, pp. 1469-1472

[4] Ma, J., *et al.*, Architectural Design of Passive Solar Residential Building, *Thermal Science, 19* (2015), 4, pp. 1415-1418

[5] He, J.-H., A New Fractal Derivation, *Thermal Science, 15* (2011), Suppl. 1, pp. S145-S147

[6] Fan, J., He, J. H., Fractal Derivative Model for Air Permeability in Hierarchic Porous Media, *Abstract and Applied Analysis*, 2012, 354701

[7] Fan, J., *et al*., Model of Moisture Diffusion in Fractal Media, *Thermal Science, 19* (2015), 4, pp. 1161-1166
[8] Hu, Y., *et al*., On Fractal Space-Time and Fractional Calculus, *Thermal Science, 20* (2016), 3, pp. 773-777
[9] Fan, J. Shang, X. M., Water Permeation in the Branching Channel net of Wool Fiber, *Heat Transfer Research, 44* (2013), 5, pp. 465-472
[10] Fan, J., Shang, X. M., Fractal Heat Transfer in Wool Fiber Hierarchy, *Heat Transfer Research, 44* (2013), 5, pp. 399-407
[11] Fei, D. D., *et al.*, Fractal Approach to Heat Transfer in Silkworm Cocoon Hierarchy, *Thermal Science, 17* (2013), 5, pp. 1546-1548
[12] Lu, T., *et al*., Analysis of Fractional Flow for Transient Two-Phase Flow in Fractal Porous Medium, *Fractals-Complex Geometry Patters and Scaling in Nature and Society, 24* (2016), 1, 1650013
[13] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[14] Liu, F. J., *et al*., He's Fractional Derivative for Heat Conduction in a Fractal Medium Arising in Silkworm Cocoon Hierarchy, *Thermal Science, 19* (2015), 4, pp. 1155-1159
[15] Liu, F. J., He, J. H., A Fractional Model for Heat-Insulating Coating with Cocoon-Like Hierarchy on Surface of Buildings, *Computers & Mathematics with Applications*, doi:10.1016/j.camwa.2016.04.018
[16] Sayevand, K., Pichaghchi, K., Analysis of Nonlinear Fractional KdV Equation Based on He's Fractional Derivative, *Nonlinear Science Letters A, 7* (2016), 3, pp. 77-85
[17] Wang, K., Liu, S., A New Solution Procedure for Nonlinear Fractional Porous Media Equation Based on a New Fractional Derivative, *Nonlinear Science Letters A, 7* (2016), 4, pp. 135-140
[18] Abbas, I. A., Youssef, H. M., Two-Dimensional Fractional Order Generalized Thermoelastic Porous Material, *Latin American Journal of Solids and Structures, 12* (2015), 7, pp. 1415-1431
[19] Fourier, J. B. J., *The Analytical Theory of Heat* (Translated by A. Freeman), Cambridge University Press, Cambridge, UK, 1878
[20] He, J. H., Lee, E. W. M., A Constrained Variational Principle for Heat Conduction, *Physics Letters A, 373* (2009), 31, pp. 2614-2615
[21] Fei, D. D., *et al*., A Short Remark on He-Lee Variational Principle for Heat Conduction, *Thermal Science, 17* (2013), 5, pp. 1561-1563
[22] Tao, Z. L., Chen, G. H., Remark on a Constrained Variational Principle for Heat Conduction, *Thermal Science, 17* (2013), 3, pp. 951-952
[23] Liu, H. Y, *et al*., A Short Remark on Stewart 1962 Variational Principle for Laminar Flow in a Uniform Duct, *Thermal Science, 20* (2106), 1, pp. 359-361
[24] Jia, Z., *et al*., Variational Principle for Unsteady Heat Conduction Equation, *Thermal Science, 1*8 (2014), 3, pp. 1045-1047
[25] Li, Z. B., Liu, J., Vartional formulations for Solton Equations Arising in Water Transport in Porous Soils, *Thermal Science, 17* (2013), 5, pp. 1483-1485
[26] Ghaneai, H., Hosseini, M. M., Variational Iteration Method with an Auxiliary Parameter for Solving Wave-Like and Heat-Like Equations in Large Domains, *Computers and Mathematics with Applications, 69* (2015), 5, pp. 363-373
[27] Liu, J. F., Modified Variational Iteration Method for Varian Boussinesq Equation, *Thermal Science, 19* (2015), 4, pp. 1195-1199
[28] Dehghan, M., *et al*., Convection-Radiation Heat Transfer in Solar Heat Exchangers Filled with a Porous Medium: Homotopy Perturbation Method *versus* Numerical Analysis, *Renewable Energy, 74* (2015), Feb., pp. 448-455
[29] Rahim-Esbo, M., *et al.*, Analytical and Numerical Investigation of Natural Convection in a Heated Cylinder Using Homotopy PerturbationMethod, *Acta Scientiarum-Technology, 36* (2014), 4, pp. 669-677
[30] Patel, T., Meher, R., Adomian Decomposition Sumudu Transform Method for Solving a Solid and Porous Fin with Temperature Dependent Internal Heat Generation, *Springer Plus, 5* (2016), Article Number489
[31] He, J. H., An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering, *Int. J. Mod. Phys. B, 22* (2008), 21, pp. 3487-3578
[32] Minea, A. A., A Review on Analytical Techniques for Natural Convection Investigation in a Heated Closed Enclosure, Case Study, *Thermal Science, 19* (2015), 3, pp. 1077-1095
[33] Sahu, S. K., *et al*., Analytical and Semi-Analytical Models of Conduction Controlled Rewetting, A State-of-Art Review, *Thermal Science, 19* (2015), 5, pp. 1479-1496

[34] He, J. H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis*, 2012, 916793
[35] He, J. H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B, 20* (2006), 10, pp. 1141-1199
[36] Tarasov, V. E., On Chain Rule for Fractional Derivatives, *Communications in Nonlinear Science and Numerical Simulation, 30* (2016), 1-3, pp. 1-4
[37] He, J. H., *et al.* Geometrical Explanation of the Fractional Complex Transform and Derivative Chain Rule for Fractional Calculus, *Phys. Lett. A, 376* (2012), pp. 257-259
[38] He, J.-H., Li, Z.-B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science, 16* (2012), 2, pp. 331-334
[39] Li, Z. B., He, J. H., Fractional Complex Transform for Fractional Differential Equations, *Math. Comput. Appl, 15* (2010), 5, pp. 970-973
[40] Zhang, M. F., *et al*., Efficient Homotopy Perturbation Method for Fractional Nonlinear Equations Using Sumudu Transform, *Thermal Science, 19* (2015), 4, pp. 1167-1171
[41] Ma, H. C., *et al*., Exact Solutions of Nonlinear Fractional Partial Differential Equations by Fractional Sub-Equation Method, *Thermal Science, 19* (2015), 4, pp. 1239-1244
[42] Hristov, J., An Approximate Analytical (Integral-Balance) Solution to a Nonlinear Heat Diffusion Equation, *Thermal Science, 19* (2015), 2, pp. 723-733
[43] Hristov, J., Approximate Solutions to Fractional Subdiffusion Equations, *European Physical Journal-Special Topics, 193* (2011), 1, pp. 229-243

Paper submitted: June 1, 2016
Paper accepted: June 6, 2016