ON THE STUDY OF VISCOUS FLUID DUE TO EXPONENTIALLY SHRINKING SHEET IN THE PRESENCE OF THERMAL RADIATION

by

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In this paper, the effect of radiation on heat transfer in boundary layer flow over an exponentially shrinking sheet is investigated analytically. The similarity transformations are used to transform the partial differential equations to ordinary ones, and an analytical solution is obtained using the homotopy perturbation method. The heat transfer characteristics for different values of the Prandtl number, Eckert number, and radiation number are analyzed and discussed. Finally, the validity of results are verified by comparing with the existing numerical results. Results are presented in tabulated forms to study the efficiency and accuracy of the homotopy perturbation method.

Key words: *boundary layer flow, heat transfer analysis, thermal radiation, exponential shrinking sheet, homotopy perturbation method*

Introduction

The dynamics of fluid flow over a stretching surface is important in many practical applications, such as extrusion of plastic sheets, paper production, glass blowing, metal spinning, and drawing plastic films [1-3]. Since the pioneering study by Sakiadis [4] on the boundary layer flow over a continuously stretching surface with a constant speed, many researchers have investigated various aspects of this problem, such as consideration of mass transfer, power-law variation of the stretching velocity and temperature, magnetic field, application to non-Newtonian fluids, and similarity solutions [5-11].

Recently, Miklavcic and Wang [12] studied the flow over a shrinking sheet. For this flow configuration, the fluid is stretched toward a slot and the flow is quite different from the stretching case. It is also shown that mass suction is required generally to maintain the flow over the shrinking sheet. In their paper, 2-D and axis-symmetric conditions were discussed and those solutions are fortunately the exact solutions of the Navier-Stokes equations. For this new type of shrinking flow, it is essentially a backward flow as discussed by Goldstein [13]. For a backward flow configuration, namely the surface moving from +1 to the slot, the fluid loses any memory of the perturbation introduced by the leading edge, say the slot. Therefore, the flow induced by a shrinking sheet shows quite distinct physical phenomena from the forward stretching flow. A search on the literature about this flow showed few publications on this subject since it is quite a new type of flow. However, heat transfer characteristics of the

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stretching/shrinking sheet problem have been restricted to two boundary conditions of either prescribed temperatures or heat flux at the wall.

Motivated by the above-mentioned investigations and applications, the current paper focuses on the 2-D boundary layer flow and heat transfer of an incompressible viscous fluid with a presence of thermal radiation over an exponentially shrinking sheet. For this purpose the governing partial differential equations are solved analytically using a highly accurate homotopy perturbation technique [14-24]. The effects of Prandtl number, Eckert number, and radiation number of viscous fluid model on the heat transfer characteristics are discussed and shown pictorially. Also our aim in this article is to compare the results given by Bidin and Nazar [11] with homotopy perturbation method (HPM) results for stretching sheet to show the efficiency and accuracy of HPM.

Mathematical model

Consider a steady two-dimensional flow of an incompressible viscous fluid over an exponentially shrinking surface. We are considering Cartesian co-ordinate system in such a way that x-axis is taken along the shrinking surface in the direction of the motion and y-axis is normal to it. The flow and heat transfer characteristics under the boundary layer approximations with the radiation effects are governed by the equations [11]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}=\nu\,\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\rho c_p\left(u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}\right)=k\,\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2-\frac{\partial q_r}{\partial y} \tag{3}$$

The corresponding boundary conditions for the flow problem are:

$$u(0)=-U_0\mathrm{e}^{\frac{x}{L}},\quad v(0)=0,\quad T(0)=T_\infty+T_0\mathrm{e}^{\frac{x}{2L}},\quad u\to 0,\quad T\to 0\ \text{as}\ y\to\infty \tag{4}$$

Using the Rosseland approximation of radiation, we can write [11]:

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \tag{5}$$

By Taylor's series we can write [11]:

$$T^4=4T_\infty^3T-3T_\infty^4 \tag{6}$$

Making use of eqs. (5) and (6), eq. (3) can be written as:

$$\rho c_p\left(u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}\right)=\left(k+\frac{16\sigma^*T_\infty^3}{3k^*}\right)\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{7}$$

Introducing the similarity variables and non-dimensional quantities [11]:

$$u = U_0 e^{\frac{x}{L}} f^{/}(\eta), \qquad v = -\sqrt{\frac{vU_0}{2L}} e^{\frac{x}{2L}} [f(\eta) + \eta f^{/}(\eta)],$$
$$T = T_\infty + T_0 e^{\frac{x}{2L}} \theta(\eta), \qquad \eta = \sqrt{\frac{U_0}{2vL}} e^{\frac{x}{2L}} y \tag{8}$$

The resulting problems can be reduced to:

$$f^{///} - 2(f^{/})^2 + ff^{//} = 0 \tag{9}$$

$$\left(1 + \frac{4}{3}K\right)\theta^{//} + \Pr[f\theta^{/} - f^{/}\theta + \mathrm{Ec}(f^{//})^2] = 0 \tag{10}$$

$$f(0) = 0, \quad f^{/}(0) = -1, \quad \theta(0) = 1, \quad f^{/} \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty \tag{11}$$

According to HPM [14-24], eqs. (9) and (10) are expressed as:

$$(1-p)L_1(f - f_0) + p[f^{///} - 2(f^{/})^2 + ff^{//}] = 0,$$
$$(1-p)L_2(\theta - \theta_0) + p\left\{\left(1 + \frac{4}{3}K\right)\theta^{//} + \Pr[f\theta^{/} - f^{/}\theta + \mathrm{Ec}(f^{//})^2]\right\} = 0 \tag{12}$$

$$f = f_0 + pf_1 + p^2 f_2 + \ldots \qquad \theta = \theta_0 + p\theta_1 + p^2\theta_2 + \ldots \tag{13}$$

Assuming $L_1 f = 0$, and $L_2\theta = 0$, and substituting f and θ from eq. (13) into eq. (12) and some simplification and re-arrangement based on powers of p-terms, we have:

$$p^{(0)} : L_1 f_0 = 0 \quad \text{and} \quad L_2\theta_0 = 0$$
$$f_0(0) = 0, \quad f_0^{/}(0) = -1, \quad \theta_0(0) = 1, \quad f_0^{/} \to 0, \quad \theta_0 \to 0 \quad \text{as} \quad \eta \to \infty \tag{14}$$

where L_1 and L_2 are defined as:

$$L_1 = \frac{\partial^3}{\partial\eta^3} + \frac{\partial^2}{\partial\eta^2} \qquad \text{and} \qquad L_2 = \frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta} \tag{15}$$

On solving eq. (14), we get initial guess:

$$f_0(\eta) = -1 + e^{-\eta} \quad \text{and} \quad \theta_0(\eta) = e^{-\eta} \tag{16}$$

$$p^{(1)} : L_1 f_1 + f_0^{///} - 2(f_0^{/})^2 + f_0 f_0^{//}) = 0, \; f_1(0) = 0, \; f_1^{/}(0) = 0, \; f_1^{/} \to 0 \; \text{as} \; \eta \to \infty$$
$$p^{(1)} : L_2\theta_1 + \left\{\left(1 + \frac{4}{3}K\right)\theta_0^{//} + \Pr[f_0\theta_0^{/} - f_0^{/}\theta_0 + \mathrm{Ec}(f_0^{//})^2]\right\} = 0 \tag{17}$$
$$\theta_1(0) = 0, \quad \theta_1 \to 0 \quad \text{as} \; \eta \to \infty$$

$$p^{(j)}: L_1 f_j - L_1 f_{j-1} + f_{j-1}^{///} - 2\left(\sum_{k=0}^{j-1} f_k^{/} f_{j-1-k}^{/}\right) + \sum_{k=0}^{j-1} f_k f_{j-1-k}^{/} = 0$$

$$f_j(0) = 0,\ f_j^{/}(0) = 0, \qquad f_j^{/} \to 0 \quad \text{as}\, \eta \to \infty$$

$$p^{(j)}: L_2\theta_j - L_2\theta_{j-1} + \tag{18}$$

$$+\left\{\left(1 + \frac{4}{3}K\right)\theta_j^{//} + \Pr\left[\sum_{k=0}^{j-1} f_k \theta_{j-1-k}^{/} - \sum_{k=0}^{j-1} f_k^{/}\theta_{j-1-k} + \text{Ec}\left(\sum_{k=0}^{j-1} f_k^{//} f_{j-1-k}^{//}\right)\right]\right\} = 0$$

$$\theta_j(0) = 0, \qquad \theta_j \to 0 \ \text{as}\, \eta \to \infty$$

Table 1. Comparison between HPM and the results obtained by Bidin and Nazar [11] for heat transfer coefficient –θ(0) for various values of Pr and *K* with Ec = 0

Pr	Bidin and Nazar [11]			HPM solution		
	Ec = 0			Ec = 0		
	$K = 0$	$K = 0.5$	$K = 1$	$K = 0$	$K = 0.5$	$K = 1$
1	0.955	0.677	0.532	0.954	0.667	0.54
2	1.471	1.074	0.863	1.48	1.071	0.866
3	1.869	1.381	1.121	1.88	1.384	1.122

Table 2. Comparison between HPM and the results obtained by Bidin and Nazar [11] for heat transfer coefficient –θ(0) for various values of Pr and *K* with Ec = 0.9

Pr	Bidin and Nazar [11]			HPM solution		
	Ec = 0.9			Ec = 0.9		
	$K = 0$	$K = 0.5$	$K = 1$	$K = 0$	$K = 0.5$	$K = 1$
1	0.539	0.410	0.334	0.541	0.419	0.335
2	0.725	0.587	0.498	0.723	0.577	0.498
3	0.830	0.696	0.605	0.833	0.689	0.605

Table 3. Comparison between HPM and the results obtained by Bidin and Nazar [11] for heat transfer coefficient –θ(0) for various values of Pr and Ec with *K* = 0

Pr	Bidin and Nazar [11]			HPM solution		
	$K = 0$			$K = 0$		
	Ec = 0	Ec = 0.2	Ec = 0.9	Ec = 0	Ec = 0.2	Ec = 0.9
1	0.955	0.862	0.538	0.954	0.863	0.541
2	1.471	1.305	0.725	1.48	1.302	0.723
3	1.870	1.688	0.830	1.88	1.687	0.833

Table 4. Comparison between HPM and the results obtained by Bidin and Nazar [11] for heat transfer coefficient –θ(0) for various values of Pr and Ec with *K* = 1

Pr	Bidin and Nazar [11]			HPM solution		
	$K = 1$			$K = 1$		
	Ec = 0	Ec = 0.2	Ec = 0.9	Ec = 0	Ec = 0.2	Ec = 0.9
1	0.532	0.488	0.334	0.54	0.487	0.335
2	0.862	0.782	0.498	0.866	0.781	0.498
3	1.121	1.007	0.605	1.122	1.005	0.605

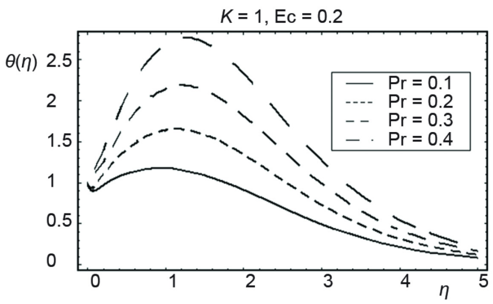

Figure 1. Effects of Pr on the temperature profiles θ

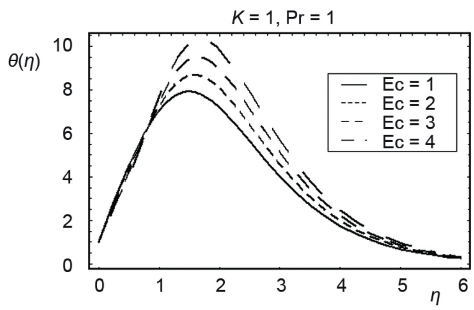

Figure 2. Effects of Ec on the temperature profiles θ

Results and discussion

Equations (14) to (18) are solved analytically using the homotopy perturbation method as described by He [14]. The graphical behavior of θ for different values of Prandtl, Eckert, and radiation numbers are presented graphically for a 10^{th} order approximation. The 10^{th} order approximation has been calculated by using Mathematica. The convergence of HPM results for a boundary layer flow over an exponentially stretching sheet [11] is shown in tabs. 1-4. An excellent agreement between the present and existing solutions [11] is achieved. Figures 1-3 elucidate the influence of Prandtl number, Eckert number, and radiation number K on the temperature profile $\theta(\eta)$. From the present study, the main results have been summarized:

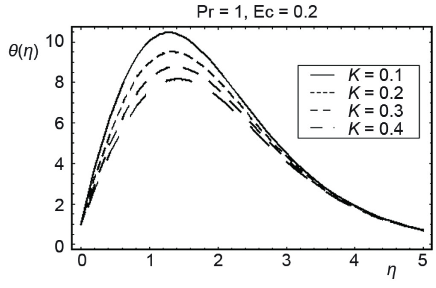

Figure 3. Effects of *K* on the temperature profiles *θ*

– the increase in the Prandtl number increases the temperature and decrease the thickness of thermal boundary layer,
– the effect of Eckert number on the temperature and thermal boundary layer thickness is quite similar to that of Prandtl number, and
– radiation number K shows opposite behavior for temperature and thermal boundary layer thickness as compared to Prandtl and Eckert numbers.

Conclusions

In this article, the viscous fluid flow and heat transfer analysis over an exponentially shrinking sheet in the presence of thermal radiation is studied. The similarity transformation is used to reduce the partial differential equations into ordinary differential equations. Analytical solutions for the velocity and temperature distribution are obtained by using the homotopy perturbation method. The results of the problem are found to be in good agreement with the existing results [11] for the stretching case. It is noted that the radiation parameter controls the thickness of thermal boundary layer.

Nomenclature

c_p – specific heat, [$Jkg^{-1}K^{-1}$]
Ec – Eckert number, [–]
f – dimensionless velocity profile, [–]
K – radiation number
k – thermal conductivity, [$Wm^{-1}K^{-1}$]
k^* – mean absorption coefficient
L – constant, [–]
Pr – Prandtl number, [–]
q_r – radiative heat flux
T – temperature, [K]
U_0 – reference velocity, [ms^{-1}]
u – velocity component in x-direction, [ms^{-1}]
v – velocity component in y-direction, [ms^{-1}]

Greek symbols

η – independent dimensionless parameter, [–]
θ – dimensionless temperature profile, [–]
μ – dynamic viscosity, [$kgm^{-1}s^{-1}$]
ρ – density of fluid, [kgm^{-3}]
σ^* – Stefan-Boltzmann constant
ν – kinematic viscosity, [m^2s^{-1}]

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