

SOLUTIONS OF FRACTIONAL DIFFUSION EQUATIONS BY VARIATION OF PARAMETERS METHOD

by

**Syed Tauseef MOHYUD-DIN^{a*}, Naveed AHMED^a, Asif WAHEED^c,
Muhammad Ali AKBAR^b, and Umar KHAN^a**

^a Faculty of Sciences, HITEC University, Taxila Cantt, Pakistan

^b Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh

^c Department of Mathematics, COMSATS Institute of Information Technology,
Attock Campus, Pakistan

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This article is devoted to establish a novel analytical solution scheme for the fractional diffusion equations. Caputo's formulation followed by the variation of parameters method has been employed to obtain the analytical solutions. Following the derived analytical scheme, solution of the fractional diffusion equation for several initial functions has been obtained. Graphs are plotted to see the physical behavior of obtained solutions.

Key words: *fractional diffusion equation, Caputo's derivative, variation of parameters method, analytical solutions*

Introduction

Fractional calculus has gained much importance in mathematical physics nowadays. There are several physical phenomena in which the fractional derivatives are involved. Plenty of literature is available that deals with the problems of fractional order [1-12] used frequently. Diffusion equations are a part of the equations that are involved in the situations of physical nature. In fractional diffusion equation, the time derivative is replaced by a fractional derivative of order α satisfying $0 < \alpha \leq 1$. Many researchers, over the years devoted significant interest to these types of problems. The fractional equation governing diffusion can be represented as:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} [F(x)u(x,t)], \quad 0 < \alpha \leq 1, \quad D > 0 \quad (1)$$

where, $\partial^\alpha(.)/\partial t^\alpha$ is the Caputo's derivative of order α [13], D – the constant, and t depends on temperature, skin friction coefficient, Avogadro number, and the universal gas constant, $u(x,t)$ is the probability density function for finding a particle at the position x in time t , and $F(x)$ is the external source. In present article, the value of D is taken to be one, $F(x) = -x$ and the fractional order is taken as $\alpha = 1/2$. Fractional diffusion equations have been tackled by number of scientists, see [5]. Inspired and motivated by the ongoing research in this area, we apply variation of parameters method (VPM) [14-16] to obtain analytical solutions for fractional diffusion equations.

* Corresponding author; e-mail: syedtauseefs@hotmail.com

Fractional diffusion model and the solution

Consider the equation:

$$\frac{\partial^{1/2}u(x,t)}{\partial t^{1/2}} = \frac{\partial^2u(x,t)}{\partial x^2} + \frac{\partial}{\partial x}[xu(x,t)] \quad (2)$$

with the initial condition $u(x,0) = f(x)$. We can also write eq. (2) in the form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial}{\partial x} [xu(x,t)] \quad (3)$$

Using the standard procedure for VPM [14-16], we can write the iterative solution scheme for eq. (3) as:

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t \lambda(s) \left\{ \frac{\partial^{1/2}}{\partial s^{1/2}} \frac{\partial^2 u(x,s)}{\partial x^2} + \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial}{\partial x} [xu(x,s)] \right\} ds \quad n \geq 1 \quad (4)$$

where, $\lambda(s)$ is the multiplier that can be obtained by using the Wronskian's technique and for the present case $\lambda(s) = 1$. $u_0(x,t)$ is the initial approximation that is obtained by using the initial condition for the problem. Taking the initial guess as $u_0(x,t) = u(x,0) = f(x)$ and using the value of multiplier, we finally have the iterative solution scheme for fractional diffusion equation as:

$$u_{n+1}(x,t) = f(x) + \int_0^t \left\{ \frac{\partial^{1/2}}{\partial s^{1/2}} \frac{\partial^2 u(x,s)}{\partial x^2} + \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial}{\partial x} [xu(x,s)] \right\} ds \quad n \geq 1 \quad (5)$$

For different values of n , various approximations of the solution can be obtained.

Illustrative examples

Example 1

First we consider $f(x) = x$. Using eq. (5), the first few iterations of the solution can be written as:

$$\begin{aligned} u_0(x,t) &= x \\ u_1(x,t) &= x + \frac{4\sqrt{tx}}{\sqrt{\pi}} \\ u_2(x,t) &= x + \frac{4\sqrt{tx}}{\sqrt{\pi}} + 4xt \\ u_3(x,t) &= x + \frac{4\sqrt{tx}}{\sqrt{\pi}} + 4xt + \frac{32x\sqrt{t^3}}{3\sqrt{\pi}} \\ u_4(x,t) &= x + \frac{4\sqrt{tx}}{\sqrt{\pi}} + 4xt + \frac{32x\sqrt{t^3}}{3\sqrt{\pi}} + 8xt^2 \\ &\vdots \end{aligned} \quad (6)$$

Finally, after 15 iterations, solution will be of the form:

$$u(x,t) = x + \frac{4\sqrt{t}x}{\sqrt{\pi}} + 4xt + \frac{32x\sqrt{t^3}}{3\sqrt{\pi}} + 8xt^2 + \frac{256x\sqrt{t^5}}{15\sqrt{\pi}} + \frac{32t^3}{3} + \frac{2048x\sqrt{t^7}}{105\sqrt{\pi}} + \dots \quad (7)$$

Graphical representations in 2-D and 3-D are plotted in figs. 1 and 2.

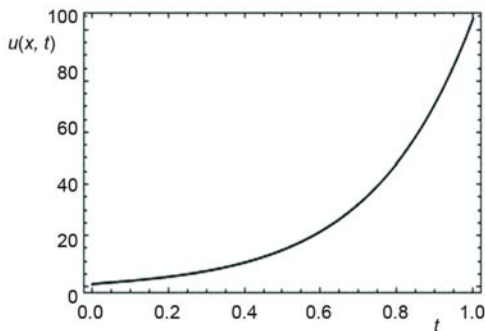


Figure 1. Variation in $u(x, t)$ for increasing t when $x = 1$ for Example 1 (2-D)

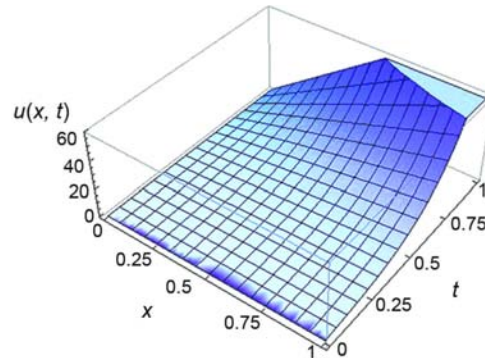


Figure 2. Variation in $u(x, t)$ for increasing t and x for Example 1 (3-D)

Example 2

Next, consider $f(x) = x^2$. Using in eq. (5), the solution can be written in the form:

$$\begin{aligned} u_0(x,t) &= x^2 \\ u_1(x,t) &= x^2 + \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{6\sqrt{t}x^2}{\sqrt{\pi}} \\ u_2(x,t) &= x^2 + \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{6\sqrt{t}x^2}{\sqrt{\pi}} + 8t + 9tx^2 \\ u_3(x,t) &= x^2 + \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{6\sqrt{t}x^2}{\sqrt{\pi}} + 8t + 9tx^2 + \frac{104\sqrt{t^3}}{3\sqrt{\pi}} + \frac{36\sqrt{t^3}x^2}{\sqrt{\pi}} \\ &\vdots \end{aligned} \quad (8)$$

and so on.

In a similar way, other terms of the solution can easily be obtained. Final form of the solution can be written as:

$$u(x,t) = x^2 + \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{6\sqrt{t}x^2}{\sqrt{\pi}} + 8t + 40t^2 + 9tx^2 + \frac{104\sqrt{t^3}}{3\sqrt{\pi}} + \frac{36\sqrt{t^3}x^2}{\sqrt{\pi}} + \frac{81t^2x^2}{2} + \dots \quad (9)$$

Graphical representations in 2-D and 3-D are plotted in figs. 3 and 4.

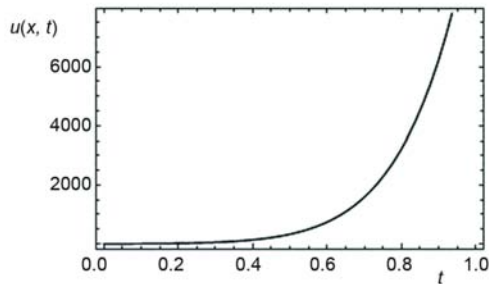


Figure 3. Variation in $u(x, t)$ for increasing t when $x = 1$ for Example 2 (2-D)

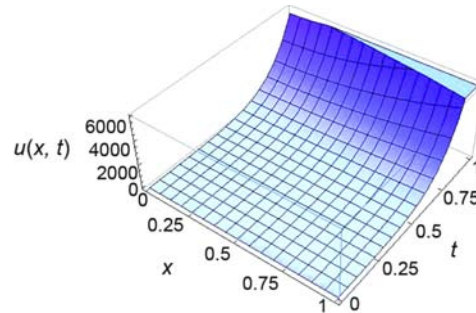


Figure 4. Variation in $u(x, t)$ for increasing t and x for Example 2 (3-D)

Example 3

Next, consider $f(x) = x^3$. Using in eq. (5), the solution can be written in the form:

$$\begin{aligned}
 u_0(x, t) &= x^3 \\
 u_1(x, t) &= x^3 + \frac{12\sqrt{t}x}{\sqrt{\pi}} + \frac{8\sqrt{t}x^3}{\sqrt{\pi}} \\
 u_2(x, t) &= x^3 + \frac{12\sqrt{t}x}{\sqrt{\pi}} + \frac{8\sqrt{t}x^3}{\sqrt{\pi}} + 36tx + 16tx^3 \\
 u_3(x, t) &= x^3 + \frac{12\sqrt{t}x}{\sqrt{\pi}} + \frac{8\sqrt{t}x^3}{\sqrt{\pi}} + \frac{672\sqrt{t^3}x}{3\sqrt{\pi}} + 16tx^3 + \frac{256\sqrt{t^3}x^3}{3\sqrt{\pi}} \\
 &\vdots
 \end{aligned} \tag{10}$$

In a similar manner, other iterations for the solution can easily be obtained.

Final form of the solution can be written as:

$$u(x, t) = x^3 + \frac{12\sqrt{t}x}{\sqrt{\pi}} + \frac{8\sqrt{t}x^3}{\sqrt{\pi}} + \frac{672\sqrt{t^3}x}{3\sqrt{\pi}} + \frac{256\sqrt{t^3}x^3}{3\sqrt{\pi}} + 16tx^3 + 360xt^2 + 128x^3t^2 + \dots \tag{11}$$

Graphical representations in 2-D and 3-D are plotted in figs. 5 and 6.

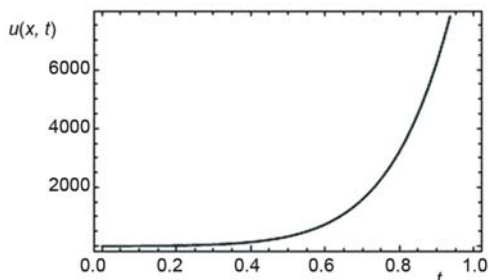


Figure 5. Variation in $u(x, t)$ for increasing t when $x = 1$ for Example 3 (2-D)

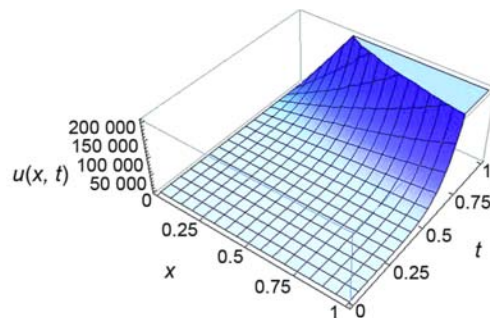


Figure 6. Variation in $u(x, t)$ for increasing t and x for Example 3 (3-D)

Example 4

Next, consider $f(x) = e^x$. Using in eq. (5), the solution can be written in the form:

$$\begin{aligned}
 u_0(x,t) &= e^x \\
 u_1(x,t) &= e^x + \frac{4e^x\sqrt{t}}{\sqrt{\pi}} + \frac{2e^x\sqrt{t}x}{\sqrt{\pi}} \\
 u_2(x,t) &= e^x + \frac{4e^x\sqrt{t}}{\sqrt{\pi}} + \frac{2e^x\sqrt{t}x}{\sqrt{\pi}} + 6e^xt + 5e^xtx + e^xtx^2 \\
 &\vdots
 \end{aligned}
 \tag{12}$$

In a similar manner, other iterations for the solution can easily be obtained. Solution in final form can be written as:

$$\begin{aligned}
 u(x,t) &= e^x + \frac{4e^x\sqrt{t}}{\sqrt{\pi}} + \frac{2e^x\sqrt{t}x}{\sqrt{\pi}} + 6e^xt + 5e^xtx + e^xtx^2 + \frac{32e^x\sqrt{t^3}}{\sqrt{\pi}} + \frac{100e^x\sqrt{t^3}x}{3\sqrt{\pi}} + \\
 &\quad + \frac{12e^x\sqrt{t^3}x^2}{\sqrt{\pi}} + \frac{4e^x\sqrt{t^3}x^3}{3\sqrt{\pi}} + \dots
 \end{aligned}
 \tag{13}$$

Graphical representations in 2-D and 3-D are plotted in figs. 7 and 8.

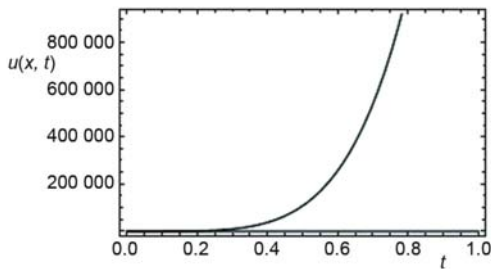


Figure 7. Variation in $u(x, t)$ for increasing t when $x = 1$ for Example 4 (2-D)

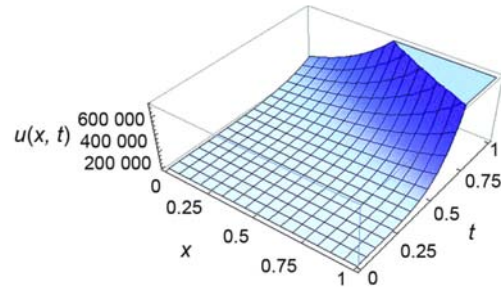


Figure 8. Variation in $u(x, t)$ for increasing t and x for Example 4 (3-D)

Example 5

Next, consider $f(x) = \sin x$. Using in eq. (5), the solution can be written in the form:

$$\begin{aligned}
 u_0(x,t) &= \sin x \\
 u_1(x,t) &= \sin x + \frac{2\cos x\sqrt{t}x}{\sqrt{\pi}} \\
 u_2(x,t) &= \sin x + \frac{2\cos x\sqrt{t}x}{\sqrt{\pi}} + xt \cos x - 2t \sin x - tx^2 \sin x - \frac{4\sqrt{t^3}x^3 \cos x}{3\sqrt{\pi}} - \\
 &\quad - \frac{20\sqrt{t^3}x \cos x}{3\sqrt{\pi}} - \frac{4\sqrt{t^3}x^2 \sin x}{\sqrt{\pi}}
 \end{aligned}$$

$$\begin{aligned}
 u_3(x,t) = & \sin x + \frac{2\cos x\sqrt{t}x}{\sqrt{\pi}} + xt \cos x - \frac{27}{2}t^2x \cos x - 3t^2x^3 \cos x - 2t \sin x - \\
 & -2t^2 \sin x - tx^2 \sin x + \frac{5}{2}t^2x^2 \sin x + \frac{1}{2}t^2x^4 \sin x - \frac{4\sqrt{t^3}x^3 \cos x}{3\sqrt{\pi}}
 \end{aligned} \quad (14)$$

and so on. Final solution after 15 iterations can be written as:

$$\begin{aligned}
 u(x,t) = & \sin x + \frac{2\cos x\sqrt{t}x}{\sqrt{\pi}} + xt \cos x - \frac{27}{2}t^2x \cos x - 3t^2x^3 \cos x - 2t \sin x - \\
 & -2t^2 \sin x - tx^2 \sin x + \frac{5}{2}t^2x^2 \sin x + \frac{1}{2}t^2x^4 \sin x - \frac{4\sqrt{t^3}x^3 \cos x}{3\sqrt{\pi}} - \\
 & -\frac{20\sqrt{t^3}x \cos x}{3\sqrt{\pi}} - \frac{4\sqrt{t^3}x^2 \sin x}{\sqrt{\pi}} - \frac{16\sqrt{t^3} \sin x}{\sqrt{\pi}} + \dots
 \end{aligned} \quad (15)$$

Graphical representations in 2-D and 3-D are plotted in figs. 9 and 10.

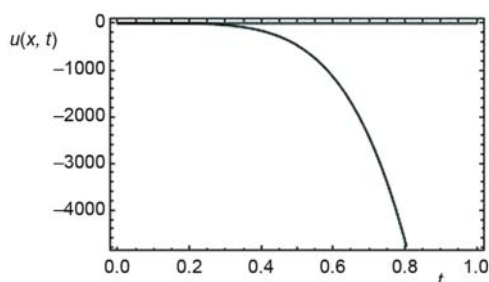


Figure 9. Variation in $u(x, t)$ for increasing t when $x = 1$ for Example 5 (2-D)

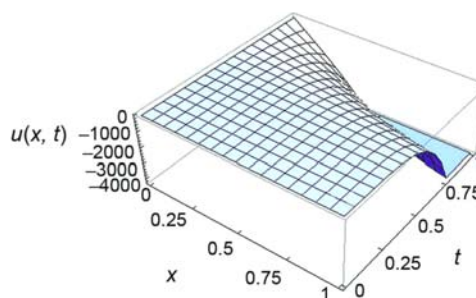


Figure 10. Variation in $u(x, t)$ for increasing t and x for Example 5 (3-D)

Numerical results and discussions

This section highlights the variations in displacement $u(x, t)$ for different values of x and t . Both the 2-D and 3-D figures are plotted for all the examples in the previous section. Figures 1-10 are displayed for the said purpose.

Figures 1 and 2 give a graphical description of Example 1. Increase in displacement $u(x, t)$ can be observed. In fig. 1, 2-D image is also provided for the case $x = 1$. Figures 3 and 4 give the same for the case of Example 2. Increase in $u(x, t)$ is quite prominent for both x and t . Similar behavior is seen in figs. 5 and 6 for the case of Example 3. From these figures, it can be concluded that the degree of polynomial for the initial guess is very important. It plays a significant role in increasing the values of $u(x, t)$ as the displacement becomes much higher for the higher powers of initial polynomials.

In figs. 7 and 8, 2-D and 3-D plots are displayed while taking the initial guess as an exponential function. For this purpose, e^x is taken as an initial guess. Since, this type of function corresponds to an exponential increase, therefore with increase in both x and t , displacement is seen to be increasing quite rapidly. This increases in much faster than that of for polynomial functions. Figures 9 and 10 are plotted to show the influence on displacement when

we take the trigonometric function as an initial guess. $\sin x$ is taken as an initial guess for this purpose. Interestingly, the displacement is seen to be a decreasing function of both x and t . This decrement is seen to be quite steep.

Conclusions

Variation of parameters method has been successfully applied used to solve the fractional problems for diffusion equation incorporating Caputo's fractional derivatives. This study can be used as a building block and can be used to solve many problems of fractional order arising in many physical phenomena.

Nomenclature

$u(x, t)$ – concentration, [–]
 x – space co-ordinate

Greek symbols

β – fractional order, [–]
 τ – time, [s]

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