SOLUTION FOR A SYSTEM OF FRACTIONAL HEAT EQUATIONS OF NANOFLUID ALONG A WEDGE

by

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Original scientific paper
DOI: 10.2298/TSCI15S1S51I

In this article, authors set a new system of fractional heat equations of nanofluid along a wedge and establish the existence and uniqueness of a solution based on the Riemann-Liouville differential operators. Sufficient conditions on the parameters of the system are imposed. A numerical solution of the system is discussed, and applications are illustrated. The technique is based on the ability of Podlubny's matrix in Matlab to formulate the operation of fractional calculus.

Key words: *fractional calculus, fractional diffusion equation, fractional heat equation*

Introduction

In the last few decades, fractional differential equations have been considered successful models of real-life phenomenon. One of the main applications of fractional calculus is modeling of intermediate physical processes. Some of the important models include fractional diffusion and wave equations. Various universal electromagnetic, mechanical, and acoustic responses can be accurately represented by utilizing fractional diffusion-wave equations [1, 5].

Different studies on fractional diffusion equations have introduced various fractional operators, such as Riemann-Liouville, Caputo, and Rize [6, 7]. Moreover, the authors imposed a maximal solution to the time-space fractional heat equation in a complex domain. Fractional time is considered in Riemann-Liouville operator, whereas fractional space is introduced in the Srivastava-Owa operator for complex variables [8].

The problem of boundary layer flow and heat transfer appearances of nanofluids has drawn much attention in recent years in response to requests for solution from the industries of medical equipment, automotive, and power plant and computer cooling systems. Since the pioneering work of Choi and Eastman [9], numerous aspects of the problem have been studied. Results of previous studies claimed that nanofluid velocity is lower than the velocity of the base fluid, and the existence of the nanofluid indicates thinning of the hydrodynamic boundary layer. Khan and Pop [10] considered numerically the boundary layer flow past a moving wedge in nanofluid and established that temperature increases with both Brownian and thermophoresis parameters. Furthermore, thermal radiation changes the temperature distribution by playing a role in controlling heat transfer process such as in polymer processing

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and nuclear reactor cooling system. Thus, much research on the effects of thermal radiation has been conducted. The study of internal heat generation or absorption is important in problems involving thermal radiation where heat may be generated or absorbed in the course of such radiation. We will show the effectiveness of these parameters in our study.

In this study, we set a new system of fractional heat equations of nanofluid along a wedge and establish the existence and uniqueness of a solution based on the Riemann-Liouville differential operators. Sufficient conditions on the parameters of the system are imposed. A numerical solution of the system is discussed, and applications are illustrated. The method is based on the ability of Podlubny's matrix in Matlab to formulate the operation of fractional calculus. Note that numerical solutions are imposed by different methods [11].

Mathematical setting

We consider the 2-D, steady, and laminar boundary layer flow over a wedge immersed in nanofluid, where the system changes in time with respect to velocity $U(t)$, temperature $T(t)$, and volume $V(t)$. The nanofluid is a diluted solid-liquid mixture with a uniform volume fraction of nanoparticle dispersed within the base fluid. The base fluid and nanoparticles are in thermally equilibrium. The effects of Brownian motion and thermophoresis are included for the nanofluid. Figure 1 shows the flow model and physical configuration. The free stream velocity of the potential flow outside the boundary layer is denoted as $G(x) = cx^m$, where c is a positive constant, m – a wedge angle parameter, $m = \beta/(2-\beta)$ and β is the Hartree pressure gradient parameter that corresponds to $\beta = \Omega/\pi$ for a total angle of the wedge. We have the following assumptions in formulating the system:

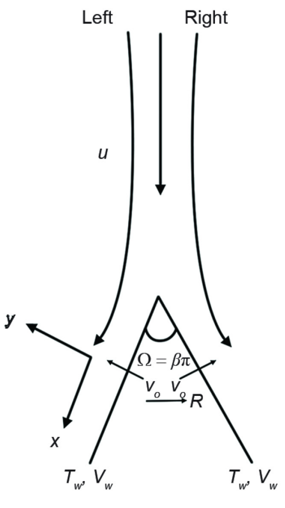

Figure 1. Flow configuration along a wedge and the co-ordinate system

– U is the two-dimensional velocity, T – the temperature of nanofluid, and V – the volume fraction of the nanoparticles $V = (V - V_\infty)/(V_w - V_\infty)$, where V_w denotes the volume at the wedge, and V_∞ denotes the volume far from the wedge with temperature T_w and T_∞, respectively.
– η_1, η_2, and η_3 are the viscosity of nanofluid, the nanofluid thermal diffusivity, and the Brownian diffusion coefficient, respectively.
– $\lambda_1 = R, \lambda_2$, and λ_3 are relaxation constants such as radiation parameter, Prandtl number and Schmited number.

None of these parameters is negative.

Taking the above assumptions into consideration, the boundary layer form of the governing equations can be written (fig. 1):

$$D_t^{\wp} U = \eta_1 \frac{\partial^2 U}{\partial x^2} + \lambda_1 \frac{\partial T}{\partial x}, \quad D_t^{\wp} T = \eta_2 \frac{\partial^2 T}{\partial x^2} + \lambda_2 \frac{\partial U}{\partial x} + G(x),$$

$$D_t^{\wp} V = \eta_3 \frac{\partial^2 V}{\partial x^2} + \lambda_3 \frac{\partial^2 T}{\partial^2 x}, \qquad \text{(1a,b,c)}$$

Fractional calculus (including fractional order indefinite integral and differentiation) is used to analyze phenomena with type $t^{\wp}$. We apply the Riemann-Liouville operators:

$$ {}_aI_t^{\wp}\upsilon(t)=\int_a^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\upsilon(\tau)\mathrm{d}\tau $$

The fractional (arbitrary) order differential of the function f of order $\wp>0$ is given by:

$$ {}_aD_t^{\wp}\upsilon(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{(t-\tau)^{-\wp}}{\Gamma(1-\wp)}\upsilon(\tau)\mathrm{d}\tau $$

We use the notation $D_t^{\wp}$ when $a=0$.

We apply the following result.

Let one of the assumptions be achieved [12]:

$$ \mu\in\eth([0,\mathrm{T}])\text{ and }\nu\in\eth^{\beta}([0,\mathrm{T}]),\quad \wp<\beta\le 1,\quad \nu\in\eth([0,\mathrm{T}])\text{ and }\mu\in\eth^{\beta}([0,\mathrm{T}]),\quad \wp<\beta\le 1 $$

$$ \mu\in\eth^{\beta}([0,\mathrm{T}])\text{ and }\nu\in\eth^{\delta}([0,\mathrm{T}]),\quad \wp<\beta\le\beta+\delta,\ \ \beta,\ \delta\in(0,1) $$

where $\eth^{\gamma}([0,\mathrm{T}])=\{\ell:[0,\mathrm{T}]\to R/\,|\,\ell(s)-\ell(s-j)|=O(j^{\gamma})$ uniformly for $0<\tau-j<s\le\mathrm{T}\}$.

Then we have:

$$ D_s^{\wp}(\ell\hbar)(s)=\ell(s)D_s^{\wp}\hbar(s)+\hbar(s)D_s^{\wp}\ell(s)- $$
$$ -\frac{\wp}{\Gamma(1-\wp)}\int_0^s\frac{[\ell(\tau)-\ell(s)][\hbar(\tau)-\hbar(s)]}{(s-\tau)^{\wp+1}}\mathrm{d}\tau-\frac{\ell(s)\hbar(s)}{\Gamma(1-\wp)s^{-\wp}} $$

point-wise.

We conclude that if μ and ν have the same sign and are both increasing or both decreasing, then:

$$ D_s^{\wp}(\ell\hbar)(s)\le\ell(s)D_t^{\wp}\hbar(s)+\hbar(s)D_s^{\wp}\ell(s) $$

and for $\ell=\hbar$, we obtain:

$$ D_s^{\wp}(\hbar^2)(s)\le 2\hbar(s)D_s^{\wp}\hbar(s) \tag{2} $$

Existence outcome

In this section, we establish global existence and uniqueness result for system (1a,b,c) subjected with the initial conditions (U_0,T_0,V_0) and the boundary condition $U(t,0)=0$, and $U(t,1)=a$, $a\ge 0$. Various solutions of thermodynamic systems are introduced in [13, 14].

Theorem 1. Assume that Ω is a bounded domain in R^2 with smooth boundary $\partial\Omega$. Consider:

$$ (U_0,T_0,V_0)\in H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega),\quad U_0>0,\ T_0\ge 0,\ V_0\ge 0\text{ in }\overline{\Omega} $$

where $H^1(\Omega)=\{\mu\in L^2(\Omega):|\nabla\mu|\in L^2(\Omega)\}$, and:

$$\left(\eta_1+\frac{\lambda_1}{2}\right)>0,\quad \left(\eta_2+\frac{\lambda_2+\lambda_3+1}{2}\right)>0,\quad \left(\eta_3+\frac{\lambda_3}{2}\right)>0$$

If:

$$\frac{\bar{\eta}\,\mathrm{T}^{\wp}}{\Gamma(\wp+1)}<1,\quad \bar{\eta}:=\max\left\{\left(\eta_1+\frac{\lambda_1}{2}\right),\left(\eta_2+\frac{\lambda_2+\lambda_3+1}{2}\right),\left(\eta_3+\frac{\lambda_3}{2}\right)\right\}$$

then there exists one bounded solution (U,T,V) for system (1).

Proof. The proof is delivered in four steps. The first three steps describe the priori estimates, whereas step 4 addresses uniqueness.

Step 1. *First estimate*. Our objective is to show that $(U, T, V) \in [L^2(\Omega), L^2(\Omega), L^2(\Omega)]$. We first expand eq. (1a) by U, utilize (2) and integrate over Ω. This approach gives the following first estimate:

$$\frac{1}{2}D_t^{\wp}\|U\|_{L^2}^2\le\eta_1\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2+\lambda_1\int_\Omega\left(\frac{\partial T}{\partial x}U\right)(t,.)\tag{3}$$

Similarly, we expand eq. (1b) and integrate over Ω to receive:

$$\frac{1}{2}D_t^{\wp}\|T\|_{L^2}^2\le\eta_2\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2+\lambda_2\int_\Omega\left(\frac{\partial U}{\partial x}T\right)(t,.)+\int_\Omega|G(x)|\tag{4}$$

Finally, we multiply eq. (1c) by V and integrate over Ω, we obtain:

$$\frac{1}{2}D_t^{\wp}\|V\|_{L^2}^2\le\eta_3\left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2+\lambda_3\int_\Omega\left(\frac{\partial^2 T}{\partial x^2}V\right)(t,.)\tag{5}$$

Collecting (3)-(5) and using the fact that U and T vanish on the boundary, we find:

$$\frac{1}{2}D_t^{\wp}(\|U\|_{L^2}^2+\|T\|_{L^2}^2+\|V\|_{L^2}^2)\le\eta_1\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2+\eta_2\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2+\eta_3\left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2+$$

$$+\lambda_1\int_\Omega\left(\frac{\partial T}{\partial x}U\right)(t,.)+\lambda_2\int_\Omega\left(\frac{\partial U}{\partial x}T\right)(t,.)+\lambda_3\int_\Omega\left(\frac{\partial^2 T}{\partial x^2}V\right)(t,.)+\int_\Omega|G(x)|\tag{6}$$

Employing the Cauchy-Schwartz inequality yields:

$$\frac{1}{2}D_t^{\wp}(\|U\|_{L^2}^2+\|T\|_{L^2}^2+\|V\|_{L^2}^2)\le\eta\left(\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2\right)+\lambda_3\left(\|V\|_{L^2}^2\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2\right)+$$

$$+\int_\Omega|G(x)|\le\lambda_3\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2(\|U\|_{L^2}^2+\|T\|_{L^2}^2+\|V\|_{L^2}^2)+\eta\left(\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2\right)+\|G\|_{L^2}\tag{7}$$

where $\eta:=\max\{\eta_1,\eta_2,\eta_3\}$. The fractional Gronwall lemma implies that:

$$\sup_{t\in[0,\mathrm{T}]}(\|U\|_{L^2}^2+\|T\|_{L^2}^2+\|V\|_{L^2}^2)\le C\varepsilon_{\wp}(\bar{\lambda}_3\mathrm{T}^{\wp})+\bar{C},\tag{8}$$

where $\varepsilon_{\wp}$ refers to the Mittag-Leffler function, $\overline{\lambda}_3 := \lambda_3 \|\partial T/\partial x\|_{L^2}^2$, C, and $\overline{C}$ are positive constants depending on $\|G\|_{L^2}$, η, λ_3, and $\wp$.

Step 2. *Second bound*. We aim to show that (U,T,V) lies in the space $[L^\infty(\Omega), L^\infty(\Omega), L^\infty(\Omega)]$. We multiply the first equation in ΔU, where ΔU is the Laplace operator and integrate over Ω, taking account that U dematerializes on the boundary of Ω, then the Cauchy-Schwartz inequality and the Young inequality yield:

$$\int_\Omega D_t^{\wp}(U\Delta U) = D_t^{\wp}\int_\Omega (U\Delta U) \le \int_\Omega \Delta U D_t^{\wp} U = \eta_1 \int_\Omega (\Delta U)^2 + \lambda_1 \int_\Omega \Delta U \frac{\partial T}{\partial x} \le$$

$$\le \eta_1 \int_\Omega (\Delta U)^2 + \lambda_1 \int_\Omega \Delta U \frac{\partial T}{\partial x} \le \eta_1 \int_\Omega (\Delta U)^2 + \lambda_1 \int_\Omega \Delta U \int_\Omega \frac{\partial T}{\partial x} \le \eta_1 \|\Delta U\|_{L^2}^2 +$$

$$+\lambda_1 \|\Delta U\|_{L^2} \left\|\frac{\partial T}{\partial x}\right\|_{L^2} \le \eta_1 \|\Delta U\|_{L^2}^2 + \frac{\lambda_1}{2}\left(\|\Delta U\|_{L^2}^2 + \left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2\right) = \left(\eta_1 + \frac{\lambda_1}{2}\right)\|\Delta U\|_{L^2}^2 + \frac{\lambda_1}{2}\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2$$

Thus, we attain to:

$$D_t^{\wp}\|\Delta U\|_{L^2}^2 \le \left(\eta_1 + \frac{\lambda_1}{2}\right)\|\Delta U\|_{L^2}^2 + \frac{\lambda_1}{2}\left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2 \tag{9}$$

In the same manner, multiplying eq. (1b) by ΔT, we obtain:

$$D_t^{\wp}\|\Delta T\|_{L^2}^2 \le \left(\eta_2 + \frac{\lambda_2 + 1}{2}\right)\|\Delta T\|_{L^2}^2 + \frac{\lambda_2}{2}\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2 + \frac{1}{2}\|G\|_{L^2}^2 \tag{10}$$

Moreover, from eq. (1c), we conclude that:

$$D_t^{\wp}\|\Delta V\|_{L^2}^2 \le \left(\eta_3 + \frac{\lambda_3}{2}\right)\|\Delta V\|_{L^2}^2 + \frac{\lambda_3}{2}\|\Delta T\|_{L^2}^2 \tag{11}$$

Joining eqs. (9)-(11) leads to:

$$D_t^{\wp}(\|\Delta U\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta V\|_{L^2}^2) \le \overline{\eta}(\|\Delta U\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta V\|_{L^2}^2) + \overline{\lambda}\left(\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2\right) +$$

$$+\frac{1}{2}\|G\|_{L^2}^2 \le \overline{\eta}(\|\Delta U\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta V\|_{L^2}^2) + \overline{\lambda}\left(\left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2\right) + \frac{1}{2}\|G\|_{L^2}^2 \tag{12}$$

where $\overline{\eta} = \max\{(\eta_1 + \lambda_1/2), [\eta_2 + (\lambda_2 + \lambda_3 + 1)/2], (\eta_3 + \lambda_3/2)\}$, $\overline{\lambda} = \max\{\lambda_1/2, \lambda_2/2, \lambda_3/2\}$.

By employing the fractional Gronwall lemma and the fact that $(U_0, T_0, V_0) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, we obtain (U,T,V) lying in the space $[L^\infty(\Omega), L^\infty(\Omega), L^\infty(\Omega)]$.

Step 3. *Third bound*. We attempt to calculate the third bound of the solution (U, T, V). Set:

$$\Phi(t) := \|\Delta U\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta V\|_{L^2}^2, \quad \Psi(t) := \left\|\frac{\partial U}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial T}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial V}{\partial x}\right\|_{L^2}^2, \quad \phi(t) := \frac{1}{2}\|G\|_{L^2}^2$$

Thus inequality (12) becomes:

$$D_t^{\wp}\Phi(t) \le \overline{\eta}\Phi(t) + \overline{\lambda}\Psi(t) + \phi(t) \tag{13}$$

Operating (13) by $I^{\wp}$, we obtain:

$$\Phi(t) \le \Phi_0 + \overline{\eta}\int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Phi(\tau)\mathrm{d}\tau + \overline{\lambda}\int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Psi(\tau)\mathrm{d}\tau + \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\phi(\tau)\mathrm{d}\tau \le$$

$$\le \Phi_0 + \frac{\overline{\eta}\,\mathrm{T}^{\wp}}{\Gamma(\wp+1)}\sup_{t\in[0,\mathrm{T}]}\Phi(t) + \frac{\overline{\lambda}\mathrm{T}^{\wp}}{\Gamma(\wp+1)}\sup_{t\in[0,\mathrm{T}]}\Psi(t) + \frac{\mathrm{T}^{\wp}}{\Gamma(\wp+1)}\sup_{t\in[0,\mathrm{T}]}\phi(t) \tag{14}$$

which implies that:

$$\sup_{t\in[0,\mathrm{T}]}\Phi(t) \le \frac{\Phi_0}{1-\dfrac{\overline{\eta}\mathrm{T}^{\wp}}{\Gamma(\wp+1)}} + \frac{\dfrac{\overline{\lambda}\mathrm{T}^{\wp}}{\Gamma(\wp+1)}}{1-\dfrac{\overline{\eta}\mathrm{T}^{\wp}}{\Gamma(\wp+1)}}\sup_{t\in[0,\mathrm{T}]}\Psi(t) + \frac{\dfrac{\mathrm{T}^{\wp}}{\Gamma(\wp+1)}}{1-\dfrac{\overline{\eta}\mathrm{T}^{\wp}}{\Gamma(\wp+1)}}\sup_{t\in[0,\mathrm{T}]}\phi(t)$$

Step 4. *Uniqueness.* Assume that (U_1, T_1, V_1) and (U_2, T_2, V_2) are two solutions for the system (1a,b,c) subjected to the initial condition:

$$(U_1^0, T_1^0, V_1^0) \in H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$$

Set:

$$(U, T, V) = (U_1 - U_2, T_1 - T_2, V_1 - V_2)$$

then we receive the system:

$$D_t^{\wp}U = \eta_1\frac{\partial^2 U}{\partial x^2} + \lambda_1\frac{\partial T}{\partial x}, \quad D_t^{\wp}T = \eta_2\frac{\partial^2 T}{\partial x^2} + \lambda_2\frac{\partial U}{\partial x}, \quad D_t^{\wp}V = \eta_3\frac{\partial^2 V}{\partial x^2} + \lambda_3\frac{\partial^2 T}{\partial^2 x} \tag{15}$$

Analogous to Step 1, we have:

$$\sup_{t\in[0,\mathrm{T}]}(\|U\|_{L^2}^2 + \|T\|_{L^2}^2 + \|V\|_{L^2}^2) \le \rho \tag{16}$$

where ρ is an arbitrary constant based on the parameters T, $\wp$ and the initial condition. Hence (1a,b,c) commit only one bounded global solution (U, T, V). This completes the proof.

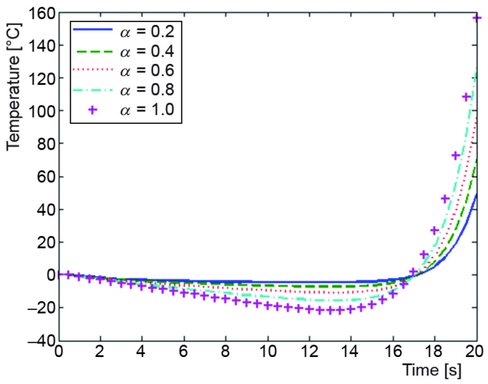

Figure 2. Temperature when m = 0, c = 10, η_2 = 1, and λ_2 = 1

Numerical results

We use a matrix method given in [15] to numerically solve system (1a,b,c). This technique is based on the ability of Podlubny's matrix in Matlab to formulate the operation of fractional calculus, which leads to the left-sided Riemann-Liouville or Caputo fractional derivative simultaneously approximated in all points of the equidistant discrimination $(j = 0, 1, ..., n)$ with the help of the upper triangular strip matrix. The Prandlt number for the base fluid is fixed in all numerical computations. In the flow over a wedge, the effect of wedge angle parameter on velocity, temperature, and nanoparticle volume fraction profiles are important. Figure 2 shows the dimensionless temperature, of nanoparticle distributions for various values of wedge angle. Indeed, decrease in the wedge angle parameter causes an increase in the heat transfer coefficient and the rate of heat transfer. The temperature profiles as the effect of suction at the wall of the wedge are demonstrated. Note that all conditions of Theorem 1 are achieved. All solutions (U, T, V) of system (1a,b,c) converge to the integer case ($\alpha = 1$). Compared with the integer system, the solutions are approximated to the steady case.

Conclusions

We conclude that the nanoparticle volume fraction profiles decrease in time as well as with velocity, whereas the temperature increases in the boundary layer flow over a wedge because of the effect of suction at the wall of the wedge.

Acknowledgments

This research is supported by Project No.: RG312-14AFR from the University of Malaya.

References

[1] Podlubny, I., *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, San Diego, Cal., USA, 1999
[2] Hilfer, R., *Applications of Fractional Calculus in Physics,* World Scientific, Singapore, 2000
[3] Kilbas, A. A., *et al., Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006
[4] Sabatier, J., *et al., Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering,* Springer, The Netherlands, 2007
[5] Lakshmikantham, V., *et al., Theory of Fractional Dynamic Systems,* Cambridge Scientific Publisher, Cambridge, UK, 2009
[6] He, J.-H., *et al.,* Converting Fractional Differential Equations into Partial Differential Equation, *Thermal Science, 16* (2012), 2 , pp. 331-334
[7] Guo, P., *et al.,* Numerical Simulation of the Fractional Langevin Equation, *Thermal Science, 16* (2012), 2, pp. 357-363
[8] Ibrahim, R. W., Jalab, H. A., Time-Space Fractional Heat Equation in the Unit Disk, *Abstract and Applied Analysis, 2013* (2013), ID 364042
[9] Choi, S. U., Eastman, J. A., Enhancing Thermal Conductivity of Fluids with Nanoparticles, *ASME Int. Mech. Eng. Congress Exposition,* San Fancisco, Cal. USA, 1995
[10] Khan, W. A., Pop, I., Boundary Layer Flow past a Wedge Moving in a Nanofluid, *Math. Probl. Eng. 2013* (2013), ID 637285
[11] Yang, X.-J., Baleanu, D., Fractal Heat Conduction Problem Solved by Local Fractional Variational Iteration Method, *Thermal Science, 17* (2013), 2, pp. 625-628
[12] Alsaedi, A., *et al.,* Maximum Principle for Certain Generalized Time and Space Fractional Diffusion Equations, *J. Quarterly Appl. Math., 73* (2015), 1, pp. 1552-4485
[13] Ran, H., Chong, C., The Thermodynamic Transitions of Antiferromagnetic Ising Model on the Fractional Multi-Branched Husimi Recursive Lattice, *Communications in Theoretical Physics, 62* (2014), 5, pp. 1-16
[14] Toure, O., Audonnet, F., Development of a Thermodynamic Model of Aqueous Solution Suited for Foods and Biological Media, Part A: Prediction of Activity Coefficients in Aqueous Mixtures Containing Electrolytes, *The Canadian Journal Chemical Engineering, 93* (2015), 2, pp. 443-450
[15] Podlubny, I., et al., Matrix Approach to Discrete Fractional Calculus II: Partial Fractional Differential Equations, *Journal of Computational Physics, 228* (2009), 8, pp. 3137-3153

Paper submitted: October 10, 2014
Paper revised: February 2, 2015
Paper accepted: February 8, 2015