

SOLUTION OF DIFFUSION EQUATION WITH LOCAL DERIVATIVE WITH NEW PARAMETER

by

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A local derivative with new parameter was used to model diffusion. The modified equation was solved iteratively. Stability of the used method together with the uniqueness of the special solution was studied. An algorithm was proposed to derive the special solution.

Key words: diffusion equation, local derivative with new parameter, stability, convergence

Introduction

In the most recent century, mathematics apparatus were employed to replicate real world problems, which take place in all branches of sciences. The diffusion equation is a partial differential equation that portrays density dynamics in a material undertakes diffusion. It is also used to describe progression demonstrating diffusive-like performance, for example the transmission of alleles in a population genetics [1-4]. The convection diffusion equation explains the flow of heat, particles, or other physical quantities in conditions where there is both diffusion and convection or advection. For information concerning the equation, its derivation, and its theoretical significance and consequences [1-4]. The following convection diffusion equation is considered here [2]:

$$c\rho \left[\frac{\partial T(x, t)}{\partial t} + \varepsilon u \frac{\partial T(x, t)}{\partial x} \right] = \lambda \frac{\partial^2 T(x, t)}{\partial x^2} + Q(x, t) \quad (1)$$

This equation can be written in the form:

$$\frac{\partial T(x, t)}{\partial t} = -\varepsilon u \frac{\partial T(x, t)}{\partial x} + a \frac{\partial^2 T(x, t)}{\partial x^2} + \frac{Q(x, t)}{c\rho} \quad (2)$$

where $a = \lambda/c\rho$ is the diffusion coefficient. One of the most used mathematical concepts in modeling is perhaps the concept of derivative. The contemporary improvement of calculus is frequently attributed to Isaac Newton and Leibniz, who provided self-determining and unified approaches to differentiation and derivatives [5-7]. Due to the complexity of the physical problems encountered in our daily basic, the concept of derivative has been modified. The concept of fractional derivative was formulated by Riemann-Liouville, and later modified by

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Caputo [8-10]. This concept has been used for modeling real world problems [11-13]. However, these derivatives do not satisfy basic properties of the Newtonian concept, for instance the product, quotient and the Chain rules that are being taught to undergraduate students. This issue has been a worry for researchers in the field of fractional calculus [14, 15]. To solve this problem, we have introduced a fractional derivative called beta-derivative. The derivative will be used in this paper to modify the diffusion equation. We shall present in the following section, the definition and some properties of the local derivative with parameter.

Definition and properties of beta-derivative

Let $a \in R$ and g be a function, such that $g : [a, \infty) \rightarrow R$. Then, the β -derivative of g is defined as:

$${}^A D_t^\beta [g(t)] = \lim_{\varepsilon \rightarrow 0} \frac{g \left\{ t + \varepsilon \left[t + \frac{1}{\Gamma(\beta)} \right]^{1-\beta} \right\} - g(t)}{\varepsilon} \quad (3)$$

for all $t \geq a$, $\beta \in (0, 1]$. The function g is said to be β -differentiable if the above limit exists.

Assume that, the given function $g : [a, \infty) \rightarrow R$ is β -differentiable at a given point $t_0 \geq a$, $\beta \in (0, 1]$, then, g is also continuous at t_0 , [16].

Assuming that f is β -differentiable on an open interval (a, b) then, [16]:

- if ${}^A D_t^\beta f(t) < 0$ for all $t \in (a, b)$ then f is decreasing there,
- if ${}^A D_t^\beta f(t) > 0$ for all $t \in (a, b)$ then f is increasing there, and
- if ${}^A D_t^\beta f(t) = 0$ for all $t \in (a, b)$ then f is constant there.

Assuming that, $g \neq 0$ and f are two functions β -differentiable with $\beta \in (0, 1]$ then, the following relations can be satisfied [16]:

$${}^A D_x^\beta [af(x) + bg(t)] = a {}^A D_t^\beta [f(t)] + b {}^A D_t^\beta [g(t)] \text{ for all } a \text{ and } b \text{ real number;}$$

$${}^A D_t^\beta (c) = 0 \text{ for } c \text{ any given constant;}$$

$${}^A D_t^\beta [f(t)g(t)] = g(t) {}^A D_t^\beta [f(t)] + f(t) {}^A D_t^\beta [g(t)];$$

$${}^A D_t^\beta \frac{f(t)}{g(t)} = \frac{g(t) {}^A D_t^\beta [f(t)] - f(t) {}^A D_t^\beta [g(t)]}{g^2(t)}$$

Assuming that $f : [a, \infty) \rightarrow R$, be a function such that f is differentiable and also α is differentiable. Let g be a function defined in the range of f and also differentiable, then we have the rule [16]:

$${}^A D_t^\beta [gof(x)] = \left(t + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} f'(t)g'[f(t)] \quad (4)$$

Let f be a function, defined in an open interval (a, b) , then the β -integral of f is given as:

$${}^A I_x^\beta [f(x)] = \int_0^x \left(t + \frac{1}{\Gamma(\beta)} \right)^{\beta-1} f(t) dt \quad (5)$$

Solution of the modified diffusion equation

One can find in the literature nowadays several methods to dealing with linear and non-linear equations. However, we shall mention the recent and efficient one, that have been intensively used, they are homotopy perturbation method [17, 18], Adomian decomposition method [19, 20], homotopy Laplace perturbation method [21], Sumudu homotopy perturbation method [22], and homotopy decomposition method [23, 24]. However, in this paper we shall use only two of these mentioned techniques namely: Laplace homotopy perturbation method, and homotopy decomposition method. Note that, the homotopy decomposition method will be used to solve the system with the β -derivative. We modify eq. (2) by replacing the local derivative with the β -derivative to obtain:

$${}_0^A D_t^\beta [T(x, t)] = -\varepsilon u \frac{\partial T(x, t)}{\partial x} + a \frac{\partial^2 T(x, t)}{\partial x^2} + \frac{Q(x, t)}{c\rho} \quad (6)$$

To solve eq. (6), we apply on both sides the inverse operator of β -derivative eq. (5) to obtain:

$$T(x, t) - T(x, 0) = {}_0^A I_t^\beta \left(-\varepsilon u \frac{\partial T(x, t)}{\partial x} + a \frac{\partial^2 T(x, t)}{\partial x^2} + \frac{Q(x, t)}{c\rho} \right) \quad (7)$$

We assume that, the solution of our equation is in the following form, with p an imbedding parameter:

$$T(x, t, p) = \sum_{k=0}^{\infty} p^k T_k(x, t) \quad (8)$$

However, replacing eq. (8) in eq. (7), we obtain:

$$\begin{aligned} & \sum_{k=0}^{\infty} p^k T_k(x, t) - T(x, 0) = \\ & = p {}_0^A I_t^\beta \left(-\varepsilon u \frac{\partial \left[\sum_{k=0}^{\infty} p^k T_k(x, t) \right]}{\partial x} + a \frac{\partial^2 \left[\sum_{k=0}^{\infty} p^k T_k(x, t) \right]}{\partial x^2} + \frac{Q(x, t)}{c\rho} \right) \end{aligned} \quad (9)$$

Now, re-arranging and putting together all terms according to their power of the embedding parameter p , we obtain the equations:

$$T_0(x, t) = T(x, 0), \quad T_1(x, t) = {}_0^A I_t^\beta \left(-\varepsilon u \frac{\partial T_0(x, t)}{\partial x} + a \frac{\partial^2 T_0(x, t)}{\partial x^2} + \frac{Q(x, t)}{c\rho} \right) \quad (10)$$

For any $k > 1$, we have the recursive formula:

$$T_k(x, t) = {}_0^A I_t^\beta \left(-\varepsilon u \frac{\partial T_{k-1}(x, t)}{\partial x} + a \frac{\partial^2 T_{k-1}(x, t)}{\partial x^2} \right)$$

It is therefore important to note that, if the first component is provided, the rest are obtained just by integration of each above equation.

Convergence analysis

One of the important parts of any iteration method is to prove the uniqueness and the convergence of the method; we are going to show the analysis underpinning the convergence and the uniqueness of the proposed method for the general solution for $p = 1$.

Assuming that X and Y are Banach spaces and $V : X \rightarrow Y$ is contraction non-linear mapping. If the progression engender by the three dimensional homotopy decomposition methods is regarded as:

$$T_n(x, t) = V[T_{n-1}(x, t)] = \sum_{k=0}^{n-1} T_k(x, t), \quad n = 1, 2, 3 \dots \quad (11)$$

Then, the following statements hold:

- $\|T_n(x, t) - T(x, t)\| \leq \rho^n \|I(x, t) - T(x, t)\|$, ($0 < \rho < 1$)
 - For any n greater than 0, $T(x, t)$ is always in the neighborhood of the exact solution $T(x, t)$ and
 - $\lim_{n \rightarrow \infty} T(x, t) = T(x, t)$
- Proof.* The proof of a) shall be achieved via induction on the natural number n . However, when $n = 1$, we obtain:

$$\|T_1(x, t) - T(x, t)\| = \|V(T_0(x, t)) - T(x, t)\|$$

However, by hypothesis, we have that V has a fixed point, which is the exact solution. Because if $T(x, t)$ is the exact solution, then:

$$T(x, t) = T_\infty(x, t) = V \left[\sum_{k=0}^{\infty-1} T_k(x, t) \right] = V \left[\sum_{k=0}^{\infty} T_k(x, t) \right] = \sum_{k=0}^{\infty} T_k(x, t)$$

since $\infty - 1$ is the same as ∞ , therefore we have that:

$$T(x, t) = V[T(x, t)]$$

Then

$$\|T_1(x, t) - T(x, t)\| = \|V[T_0(x, t)] - V[T(x, t)]\|$$

$$\|V[T_0(x, t)] - V[T(x, t)]\| = \|V[T_{n-1}(x, t)] - T(x, t)\| = \|V[T_{n-1}(x, t)] - V[T(x, t)]\|$$

Using the fact that V is a non-linear contractive mapping we have the inequality:

$$\|V[T_0(x, t)] - V[T(x, t)]\| < \rho \|T_{n-1}(x, t) - V[T(x, t)]\|$$

Furthermore using the induction hypothesis, we arrive at:

$$\rho \|T_{n-1}(x, t) - V[T(x, t)]\| < \rho \rho^{n-1} \|T_0(x, t) - T(x, t)\|$$

and the proof is completed.

Again we shall proof this by employing induction technique on m . For $m = 0$, we have:

$$T_0(x, t) = I(x, t) = \sum_{w=0}^{n-1} \sum_{h=0}^{m-1} \frac{f_{w,h}}{w!h!}$$

According to the idea of the homotopy decomposition method, previous equation is the contribution of the initial conditions. More importantly, the above is nothing more than Taylor series of the exact solution of order nml , thus this leads us to the situation that, we can find a positive real number r such that:

$$\|T_0(x, t) - T(x, t)\| < r$$

This is true, because the contribution of the initial conditions is in the same neighborhood of the exact solution. Then the property is verified for $m = 0$, let us assume that, the property is also true for $m - 1$, that is we assume that, we can find a positive real number r such that:

$$\|T_{n-1}(x, t) - T(x, t)\| < r$$

We now want to show that the property is also true for m . In fact:

$$\|T_{n-1}(x, t) - T(x, t)\| = \|V[T_{n-1}(x, t)] - V[T(x, t)]\|$$

using the fact that V is a non-linear contractive mapping leads us to obtain:

$$\|V[T_{n-1}(x, t)] - V[T(x, t)]\| = \|V[T_{n-1}(x, t)] - V[T(x, t)]\| < \rho r$$

Since $\rho < 1$, we finally have:

$$\|T_m(x, t) - T(x, t)\| < r$$

and this completes the proof.

The proof of c) is directly achieved using the a) according to:

$$\lim_{n \rightarrow \infty} \|T_n(x, t) - T(x, t)\| \leq \lim_{n \rightarrow \infty} \rho^n \|I(x, t) - T(x, t)\| = 0$$

Then

$$\lim_{n \rightarrow \infty} T_n(x, t) = T(x, t)$$

Uniqueness analysis

To partially show the efficiency of the used method, we present in this section the uniqueness of the special solution for using the above technique. To achieve this, we assume that the exact solution of eq. (6) exists and that, the special solution converge to the exact solution for larger natural number N . Let assume by contradiction that, there exist two different special solutions $T_1(x, t)$ and $T_2(x, t)$ then consider the Hilbert space:

$$H = \frac{T_1, T_2}{\int T_1, T_2 < \infty} \tag{12}$$

Proof. Also consider the following operator:

$$P[T(x, t)] = {}^A_0 D_t^\beta [T(x, t)] = -\varepsilon u \frac{\partial T(x, t)}{\partial x} + a \frac{\partial^2 T(x, t)}{\partial x^2} + \frac{Q(x, t)}{c\rho} \tag{13}$$

The aim of our proof is to show, using the inner product that:

$$|T_2 - T_1| < \varepsilon \tag{14}$$

To achieve this, we evaluate:

$$(P(T_2) - P(T_1), D) \quad \text{for} \quad D \in H = \frac{u, v}{\int uv < \infty} \quad (15)$$

However

$$P(T_2) - P(T_1) = -\varepsilon u \frac{\partial [T_2(x, t) - T_1(x, t)]}{\partial x} + a \frac{\partial^2 [T_2(x, t) - T_1(x, t)]}{\partial x^2} \quad (16)$$

Thus

$$(P(T_2) - P(T_1), D) = \left(\varepsilon u \frac{\partial (T_1 - T_2)}{\partial x}, D \right) + \left(a \frac{\partial^2 (T_2 - T_1)}{\partial x^2}, D \right) \quad (17)$$

We shall evaluate case by case, thus we start with the first component:

$$\left(\varepsilon u \frac{\partial (T_1 - T_2)}{\partial x}, D \right)$$

But using the Schwartz inequality, we obtain:

$$\left(\varepsilon u \frac{\partial (T_1 - T_2)}{\partial x}, D \right) < |D| \|(T_1 - T_2)_x| \varepsilon u \quad (18)$$

But using the properties of the inner product and using the continuity of the partial derivative, we can find a positive constant w such that:

$$\left(\varepsilon u \frac{\partial (T_1 - T_2)}{\partial x}, D \right) < w |D| \|(T_1 - T_2)| \varepsilon u \quad (19)$$

Using the same routine, we obtain the inequality for the second component:

$$\left(a \frac{\partial^2 (T_2 - T_1)}{\partial x^2}, D \right) < a O_1 O_2 |D| \|(T_2 - T_1)| \quad (20)$$

where O_1 and O_2 are positive constants. Now putting eqs. (19) and (20) into eq. (17) to obtain:

$$(P(T_2) - P(T_1), D) < (w\varepsilon + aO_1O_2) |D| \|(T_2 - T_1)| \quad (21)$$

Now since T is the exact solution of our modified equation, then T_1 and T_2 converge to T , thus, we can find two large natural number N and M such that:

$$|T - T_1| \ll \frac{\varepsilon}{2(w\varepsilon + aO_1O_2) |D|} \quad \text{for } N \quad (22)$$

$$|T_2 - T| \ll \frac{\varepsilon}{2(w\varepsilon + aO_1O_2) |D|} \quad \text{for } M \quad (23)$$

Now take $m = \max(N, M)$, then:

$$|T_2 - T_1| \ll \frac{\varepsilon}{2(w\varepsilon + aO_1O_2) |D|} \quad \text{for } M \quad (25)$$

Replacing eq. (25) in eq. (21), we arrive at:

$$(P(T_2) - P(T_1), D) \ll \varepsilon \quad (26)$$

With ε extremely very small, we have that $|T_2 - T_1| = 0$, this implies $T_2 = T_1$.

Conclusion

A novel local derivative was used in this work to model the convection-diffusion. This derivative is a local derivative with fractional order. Some useful properties of the new derivative were presented. The modified equation was solved iteratively using the well known technique homotopy decomposition method, which is the modification of homotopy perturbation method. To show the efficiency of the used method, we presented a detail analysis of the stability and the uniqueness of the special solution.

Nomenclature

c – specific heat, [$\text{Jkg}^{-1}\text{K}^{-1}$]
 $Q(x, t)$ – source term, [Wm^{-3}]
 T – temperature, [K]
 t – time, [s]
 u – velocity, [ms^{-1}]

Greek symbols

ε – porosity, [-]
 λ – thermal conductivity, [$\text{WK}^{-1}\text{m}^{-1}$]
 ρ – mass density, [kg]

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