A NEW COMPUTATIONAL METHOD FOR THE ONE-DIMENSIONAL DIFFUSION PROBLEM WITH THE DIFFUSIVE PARAMETER VARIABLE IN FRACTAL MEDIA

by

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In this paper, we use a local fractional Laplace decomposition method to solve the diffusion equation with the diffusive parameter variable in fractal media. The obtained result illustrates the efficiency and accuracy of the proposed method to obtain analytical solutions to differential equations within the local fractional derivatives.

Key words: *fractional diffusion equation, Laplace transform, Adomian decomposition method, local fractional derivative*

Introduction

The theory of local fractional calculus has played an important role in the areas ranging from fundamental science to engineering in the past ten years (see, [1-3], and the references therein). It has been applied to a wide class of complex problems encompassing physics [4-7], mechanics [8-12] and interdisciplinary areas [13-15]. The DM [16], FM [16-18], HPM [19], VIM [20-22], FT [23, 24], LT [25-27], LDM [28], and ST [29] *via* local fractional derivatives (LFD) have been utilized to solve local fractional differential equations.

The diffusion equation (called LFDE) in fractal media was recently described in [17] as:

$$\frac{\partial^{\nu} H(\mu,\tau)}{\partial \tau^{\nu}} = \eta^{2\nu} \frac{\partial^{2\nu} H(\mu,\tau)}{\partial \mu^{2\nu}} \tag{1}$$

where $\eta^{2\nu}$ is the fractal diffusion parameter, and $H(\mu,\tau)$ – a non-differentiable function. The present work deals with a compact solution to diffusion equation on Cantor sets by using the local fractional Laplace decomposition method (LFLDM) [28].

Preliminaries

It is so-called local fractional continuity in the interval (σ,υ), denoted by $\Theta(\mu) \in C_{\nu}(\sigma,\upsilon)$.

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Setting $\Theta(\mu)\in C_{\nu}(\sigma,\upsilon)$, the LFD of $\Theta(\mu)$ of order ν at $\mu=\mu_0$ is given as [1-3, 11-26]:

$$D_{\mu}^{\nu}\Theta(\mu_0)=\Theta^{(\nu)}(\mu_0)=\lim_{\mu\to\mu_0}\frac{\Delta^{\nu}[\Theta(\mu)-\Theta(\mu_0)]}{(\mu-\mu_0)^{\nu}} \tag{2}$$

where $\Delta^{\nu}[\Theta(\mu)-\Theta(\mu_0)]\cong\Gamma(\nu+1)[\Theta(\mu)-\Theta(\mu_0)]$.

A partition of the interval (σ,υ) is denoted as (μ_j,μ_{j+1}), $j=0,...,N-1$, $\mu_0=\sigma$ and $\mu_N=\upsilon$ with $\Delta\mu_j=\mu_{j+1}-\mu_j$ and $\Delta\mu=\max\{\Delta\mu_0,\Delta\mu_1,.....\}$. Setting $\Theta(\mu)\in C_{\nu}(\sigma,\upsilon)$, the local fractional integral operator of $\Theta(\mu)$ in the interval (σ,υ) is given as below [1-3, 27-29]:

$${}_{\sigma}I_{\upsilon}^{(\nu)}\Theta(\mu)=\frac{1}{\Gamma(1+\nu)}\int_{\sigma}^{\upsilon}\Theta(\mu)(\mathrm{d}\mu)^{\nu}=\frac{1}{\Gamma(1+\nu)}\lim_{\Delta\mu\to 0}\sum_{j=0}^{N-1}\Theta(\mu_j)(\Delta\mu_j)^{\nu} \tag{3}$$

Set

$$\frac{1}{\Gamma(1+\nu)}\int_{0}^{\infty}\left|\Theta(\mu)\right|(\mathrm{d}\mu)^{\nu}<k<\infty \tag{4}$$

The local fractional Laplace transform (LFLT) of $\Theta(\mu)$ is given as [25-27]:

$$L_{\nu}\{\Theta(\mu)\}=\Theta_{s}^{L,\nu}(s)=\frac{1}{\Gamma(1+\nu)}\int_{0}^{\infty}E_{\nu}(-s^{\nu}\mu^{\nu})\Theta(\mu)(\mathrm{d}\mu)^{\nu},\quad 0<\nu\le 1 \tag{5}$$

where the latter integral converges and $s^{\nu}\in R^{\nu}$.

Its inverse formula is given as [2, 3, 25-27]:

$$L_{\nu}^{-1}\{\Theta_{s}^{L,\nu}(s)\}=\Theta(\mu)=\frac{1}{(2\pi)^{\nu}}\int_{\beta-i\omega}^{\beta+i\omega}E_{\nu}(s^{\nu}\mu^{\nu})\Theta_{s}^{L,\nu}(s)(\mathrm{d}s)^{\nu},\quad 0<\nu\le 1 \tag{6}$$

where $s^{\nu}=\beta^{\nu}+i^{\nu}\omega^{\nu}$, the fractal imaginary unit i^{ν} and $\mathrm{Re}(s)=\beta>0$.

The properties of local fractional Laplace transform are listed as [2, 3, 25-27]:

$$L_{\nu}\{\Theta^{(k\nu)}(\mu)\}=s^{k\nu}\Theta_{s}^{L,\nu}(s)-s^{(k-1)\nu}\Theta(0)-s^{(k-2)\nu}\Theta^{(\nu)}(0)-\cdots-\Theta^{[(k-1)\nu]}(0) \tag{7}$$

$$L_{\nu}\{E_{\nu}(c^{\nu}\mu^{\nu})\Theta(\mu)\}=\Theta_{s}^{L,\nu}(s-c) \tag{8}$$

$$L_{\nu}\{E_{\nu}(a^{\nu}\mu^{\nu})\}=\frac{1}{s^{\nu}-a^{\nu}} \tag{9}$$

$$L_{\nu}\{\sin_{\nu}(a^{\nu}\mu^{\nu})\}=\frac{a^{\nu}}{s^{2\nu}+a^{2\nu}} \tag{10}$$

$$L_{\nu}\left\{\frac{\mu^{k\nu}}{\Gamma(1+k\nu)}\right\}=\frac{1}{s^{(k+1)\nu}} \tag{11}$$

Analysis of the LFDM

We now rewrite (1) in local fractional differential operator form:

$$F_\nu H(\mu,\tau)+R_\nu\, H(\mu,\tau)=0 \tag{12}$$

where $F_\nu=\partial^\nu/\partial\tau^\nu$, and $R_\nu=-\eta^{2\nu}\partial^{2\nu}/\partial\mu^{2\nu}$.
Taking the LFLT on eq. (12), we obtain:

$$L_\nu\{F_\nu H(\mu,\tau)\}+L_\nu\{R_\nu H(\mu,\tau)\}=0 \tag{13}$$

Adopting (7), we have:

$$s^\nu L_\nu\{H(\mu,\tau)\}-H(\mu,0)+L_\nu\{R_\alpha H(\mu,\tau)\}=0 \tag{14}$$

or

$$L_\nu\{H(\mu,\tau)\}=\frac{1}{s^\nu}H(\mu,0)-\frac{1}{s^\nu}L_\nu\{R_\nu H(\mu,\tau)\} \tag{15}$$

Taking the inverse of the LFLT on (15), we get:

$$H(\mu,\tau)=H(\mu,0)-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\{R_\nu H(\mu,\tau)\}\right] \tag{16}$$

We are going to represent the solution in an infinite series:

$$H(\mu,\tau)=\sum_{n=0}^{\infty}H_n(\mu,\tau) \tag{17}$$

Substituting (20) into (19), which give us this result:

$$\sum_{n=0}^{\infty}H_n(\mu,\tau)=H(\mu,0)-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\left\{R_\nu\sum_{n=0}^{\infty}H_n(\mu,\tau)\right\}\right] \tag{18}$$

Comparing the left and right hand sides of (18) we obtain:

$$H_0(\mu,\tau)=H(\mu,0) \tag{19}$$

$$H_1(\mu,\tau)=-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\{R_\nu H_0(\mu,\tau)\}\right] \tag{20}$$

$$H_2(\mu,\tau)=-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\{R_\nu H_1(\mu,\tau)\}\right] \tag{21}$$

$$H_3(\mu,\tau)=-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\{R_\nu H_2(\mu,\tau)\}\right] \tag{22}$$

and so on.

The recursive relations, in general form are:

$$H_0(\mu,\tau)=H(\mu,0) \tag{23}$$

$$H_n(\mu,\tau)=-L_\nu^{-1}\left[\frac{1}{s^\nu}L_\nu\{R_\nu H_{n-1}(\mu,\tau)\}\right] \tag{24}$$

The approximate solution with non-differentiable terms

We present an initial condition $H(\mu,0)=\mu^{2\nu}/\Gamma(1+2\nu)$, and $\eta^{2\nu}=E_\nu(\mu^\nu)$.

In view of (23) and (24) the local fractional iteration algorithm can be written as:

$$H_0(\mu,\tau)=\frac{\mu^{2\nu}}{\Gamma(1+2\nu)} \tag{25}$$

$$H_{n+1}(\mu,\tau)=L_\nu^{-1}\left[\frac{1}{s^\alpha}E_\nu(\mu^\nu)L_\nu\left\{\frac{\partial^{2\nu}H_n(\mu,\tau)}{\partial\mu^{2\nu}}\right\}\right],\quad n\ge 0 \tag{26}$$

Therefore, from (26) we give the components:

$$H_0(\mu,\tau)=\frac{\mu^{2\nu}}{\Gamma(1+2\nu)} \tag{27}$$

$$H_1(\mu,\tau)=L_\nu^{-1}\left[\frac{1}{s^\nu}E_\nu(\mu^\nu)L_\nu\left\{\frac{\partial^{2\nu}H_0(\mu,\tau)}{\partial\mu^{2\nu}}\right\}\right]=L_\nu\left[\frac{1}{s^{2\nu}}E_\nu(\mu^\nu)\right]=\frac{\tau^\nu}{\Gamma(1+\nu)}E_\nu(\mu^\nu) \tag{28}$$

$$H_2(\mu,\tau)=L_\nu^{-1}\left[\frac{1}{s^\nu}E_\nu(\mu^\nu)L_\nu\left\{\frac{\partial^{2\nu}H_1(\mu,\tau)}{\partial\mu^{2\nu}}\right\}\right]=L_\nu\left[\frac{1}{s^{3\nu}}E_\nu(\mu^\nu)\right]=\frac{\tau^{2\nu}}{\Gamma(1+2\nu)}E_\nu(\mu^\nu) \tag{29}$$

$$H_3(\mu,\tau)=L_\nu^{-1}\left[\frac{1}{s^\nu}\frac{\mu^{2\nu}}{\Gamma(1+2\nu)}L_\nu\left\{\frac{\partial^{2\nu}H_1(\mu,\tau)}{\partial\mu^{2\nu}}\right\}\right]=L_\nu\left[\frac{1}{s^{4\nu}}E_\nu(\mu^\nu)\right]=\frac{\tau^{3\nu}}{\Gamma(1+3\nu)}E_\nu(\mu^\nu) \tag{30}$$

and so on.

Consequently, we obtain:

$$H(\mu,\tau)=\sum_{n=0}^{\infty}H_n(\mu,\tau)=$$

$$=E_\nu(\mu^\nu)E_\nu(\tau^\nu)+\frac{\mu^{2\nu}}{\Gamma(1+2\nu)}-E_\nu(\mu^\nu) \tag{31}$$

and its graph with $\nu=\ln 2/\ln 3$ is illustrated in fig. 1.

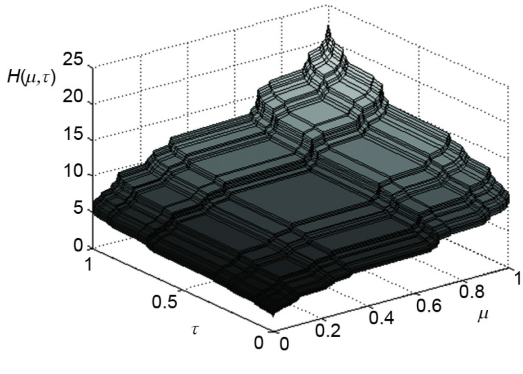

Figure 1. The one-dimensional diffusion problem with the diffusive parameter variable $\eta^{2\nu}=E_\nu(\mu^\nu)$ when ν = ln2/ln3

Conclusions

The LFDM is demonstrated as an effective technology for solutions of a wide class of local fractional differential equations. The analytical solution of the diffusion equation with the diffusive parameter variable is successfully developed by recurrence relations resulting in convergent series solutions.

Nomenclature

$H(\eta, \tau)$ – concentration, [–]
$L_\nu\{\Theta(\mu)\}$ – LFLT
$L_\nu^{-1}\{\Theta_s^{L,\nu}(s)\}$ – the inverse formula of LFLT

Greek symbols

μ – space co-ordinates, [m]
ν – time fractal dimensional order, [–]
τ – time, [s]

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