EXACT SOLUTIONS OF NON-LINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY FRACTIONAL SUB-EQUATION METHOD

by

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This paper studies the space-time fractional Whitham-Broer-Kaup equations by the existed fractional sub-equation method, and exact solutions are obtained.

Key words: *fractional partial differential equations, exact solutions, the fractional sub-equation method*

Introduction

Fractional partial differential equations are generalization of the classical differential equations of integer order. In recent decades, fractional differential equations have gained a lot of attention as they are widely used to describe a variety of complex phenomena in many fields [1-3]. In the past, many powerful methods were established and developed to obtain exact solutions and numerical solutions of the fractional differential equation (FDE), such as the finite difference method [4], the Adomian decomposition method [5], and so on.

In this paper, we use the existed fractional sub-equation method to search for exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations in the sense of modified Riemann-Liouville derivative defined by Jumarie [6], which is a fractional version of the known (G'/G) method [7]. This method is based on the following fractional ODE:

$$D_t^{2\alpha} G(\xi) + \lambda D_t^{\alpha} G(\xi) + \mu G(\xi) = 0 \tag{1}$$

Jumarie's modified Riemann-Liouville derivative and existed fractional sub-equation method

We list some important properties for the modified Riemann-Liouville derivative [6]:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha} \tag{2}$$

$$D_t^{\alpha}[f(t)g(t)] = g(t)D_t^{\alpha} f(t) + f(t)D_t^{\alpha} g(t) \tag{3}$$

$$D_t^{\alpha} f[g(t)] = f_g^{'}[g(t)]D_t^{\alpha} g(t) = D_t^{\alpha} f[g(t)][g'(t)]^{\alpha} \tag{4}$$

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In order to obtain the general solutions for eq. (1), we suppose $G(\xi) = H(\eta)$ and use the well-known fractional complex transformation [8], $\eta = \xi^\alpha/\Gamma(1 + \alpha)$. Then by using eq. (2) and the first equality in eq. (4) and eq. (1) can be turned into the following second ODE:

$$H''(\eta) + \lambda H'(\eta) + \mu H(\eta) = 0$$

since $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta)\, D_\xi^\alpha \eta = H'(\eta)$, we obtain:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} -\dfrac{\lambda}{2} + \dfrac{\sqrt{\lambda^2 - 4\mu}}{2} \dfrac{C_1 \sinh \dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}, & \lambda^2 - 4\mu > 0 \\ -\dfrac{\lambda}{2} + \dfrac{C_2 \Gamma(1+\alpha)}{C_1\Gamma(1+\alpha) + C_2 \xi^\alpha}, & \lambda^2 - 4\mu = 0 \\ -\dfrac{\lambda}{2} + \dfrac{\sqrt{4\mu - \lambda^2}}{2} \dfrac{-C_1 \sin \dfrac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \dfrac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \dfrac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sin \dfrac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}, & \lambda^2 - 4\mu < 0 \end{cases} \quad (5)$$

Description of the existed fractional sub-equation method

In this section, we describe the main steps of the existed fractional sub-equation method.

Step 1. Suppose that a non-linear FDE, say in two independent variables x and t:

$$P(u, u_t, u_x, D_t^\alpha u, D_x^\alpha u, \cdots) = 0, \qquad 0 < \alpha \le 1 \quad (6)$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann-Liouville derivatives of u, $u = u(x, t)$ is an unknown function, P – a polynomial in u, and its various partial derivatives, in which the highest order derivatives and non-linear terms are involved.

Step 2. By using the traveling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = x + ct + \xi_0 \quad (7)$$

then, by the second equality in eq. (4) and eq. (6) can be turned into the following fractional ODE with respect to the variable ξ:

$$\tilde{P}(u, cu', u', c^\alpha D_\xi^\alpha u, D_\xi^\alpha u, \cdots) = 0 \quad (8)$$

Step 3. Suppose that the solution of eq. (8) can be expressed by a polynomial in $D_\xi^\alpha G/G$:

$$u(\xi) = \sum_{i=0}^{m} a_i \frac{D_\xi^\alpha G^i}{G} \quad (9)$$

where $G = G(\xi)$ satisfies eq. (1), and $a_i (i = 0, 1, \ldots, m)$ are constants to be determined later with $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and non-linear terms appearing in eq. (8).

Step 4. Substituting eq. (9) into eq. (8), using eq. (1) and collecting all terms with the same order of $D_\xi^\alpha G/G$ together, the left-hand side of eq. (8) is converted into another polynomial in $D_\xi^\alpha G/G$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_i (i = 0, 1, \ldots, m)$.

Step 5. Solving the equation system in Step 4 and using eq. (5), we can construct a variety of exact solutions for eq. (6).

Applications

The space-time fractional WBK equations:

$$\begin{cases} D_t^\alpha u + u D_x^\alpha u + D_x^\alpha v + \beta D_x^{2\alpha} u = 0 \\ D_t^\alpha v + D_x^\alpha (uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0 \end{cases} \tag{10}$$

can be used to describe the dispersive long wave in shallow water. Here $u = u(x, t)$ is the field of horizontal velocity, $v = v(x, t)$ is the height deviating from equilibrium position of liquid, β and γ are real constants that represent different diffusion powers. When $\alpha = 1$, $\beta = 0$, and $\gamma = 0$, eq. (10) is the classical long-wave equations that describe the shallow water wave with diffusion. When $\alpha = 1$, $\beta = 0$, and $\gamma = 1$, eq. (10) reduces to the variant Boussinesq equations [9] which are very important in fluid mechanics.

Suppose that $u = u(x, t)$, $v = v(x, t)$, where $\xi = x + ct + \xi_0$, k, c, ξ_0 are all constants with $k, c \neq 0$. Then by use of the second equality in eq. (4) and eq. (10) can be turned into:

$$\begin{cases} c^\alpha D_\xi^\alpha u + k^\alpha u D_\xi^\alpha u + k^\alpha D_\xi^\alpha v + k^{2\alpha} \beta D_\xi^{2\alpha} u = 0 \\ c^\alpha D_\xi^\alpha v + k^\alpha D_\xi^\alpha (uv) + k^{2\alpha} \beta D_\xi^{2\alpha} v + k^{3\alpha} \gamma D_\xi^{3\alpha} u = 0 \end{cases} \tag{11}$$

Assume that the solution of eq. (11) can be expressed by:

$$\begin{cases} u(\xi) = \displaystyle\sum_{i=0}^{m_1} a_i \left(\frac{D_\xi^\alpha G}{G} \right)^i \\ v(\xi) = \displaystyle\sum_{i=0}^{m_2} b_i \left(\frac{D_\xi^\alpha G}{G} \right)^i \end{cases} \tag{12}$$

Balancing the order of $D_\xi^{2\alpha} u$, $u D_\xi^\alpha u$, $D_\xi^{3\alpha} u$, and $D_\xi^\alpha (uv)$ in eq. (11), we can obtain $m_1 = 1$, and $m_2 = 2$.

We have:

$$\begin{cases} u(\xi) = a_0 + a_1 \dfrac{D_\xi^\alpha G}{G} \\ v(\xi) = b_0 + b_1 \dfrac{D_\xi^\alpha G}{G} + b_2 \left(\dfrac{D_\xi^\alpha G}{G} \right)^2 \end{cases} \tag{13}$$

Substituting eq. (13) into eq. (11), using eq. (1) and collecting all the terms with the same power of $D_\xi^\alpha G/G$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields:

$$\begin{cases} a_0 = -\dfrac{c^\alpha}{k^\alpha} \pm k^\alpha \lambda\sqrt{\beta^2+\gamma} \\ a_1 = \pm 2k^\alpha\sqrt{\beta^2+\gamma} \\ b_0 = k^{2\alpha}\left(\pm 2\beta\mu\sqrt{\beta^2+\gamma} - 2\beta^2\mu - 2\mu\gamma\right) \\ b_1 = k^{2\alpha}\left(\pm 2\beta\lambda\sqrt{\beta^2+\gamma} - 2\beta^2\lambda - 2\lambda\gamma\right) \\ b_2 = k^{2\alpha}\left(\pm 2\beta\sqrt{\beta^2+\gamma} - 2\beta^2 - 2\gamma\right) \end{cases} \tag{14}$$

Substituting eq. (14) into eq. (13) and combining with eq. (5), we can obtain the exact solutions of eq. (10).

When $\lambda^2 - 4\mu = 0$, we have:

$$\begin{cases} u_3(x,t) = -\dfrac{c^\alpha}{k^\alpha} \pm k^\alpha \lambda\sqrt{\beta^2+\gamma} \pm 2k^\alpha\sqrt{\beta^2+\gamma}\left[-\dfrac{\lambda}{2} + \dfrac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] \\ v_3(x,t) = k^{2\alpha}\left(\pm 2\beta\mu\sqrt{\beta^2+\gamma} - 2\beta^2\mu - 2\mu\gamma\right) + \\ \qquad + k^{2\alpha}\left(\pm 2\beta\lambda\sqrt{\beta^2+\gamma} - 2\beta^2\lambda - 2\lambda\gamma\right)\left[-\dfrac{\lambda}{2} + \dfrac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right] + \\ \qquad + k^{2\alpha}\left(\pm 2\beta\sqrt{\beta^2+\gamma} - 2\beta^2 - 2\gamma\right)\left[-\dfrac{\lambda}{2} + \dfrac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\right]^2 \end{cases} \tag{15}$$

where $\xi = kx + ct + \xi_0$.

When $\lambda^2 - 4\mu > 0$:

$$u_1(x,t) = -\frac{c^\alpha}{k^\alpha} \pm k^\alpha \lambda\sqrt{\beta^2+\gamma} \pm$$

$$\pm 2k^\alpha\sqrt{\beta^2+\gamma}\left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{C_1\sinh\dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cosh\dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\sinh\dfrac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\} \tag{16}$$

$$v_1(x,t)=k^{2\alpha}\left(\pm 2\beta\mu\sqrt{\beta^2+\gamma}-2\beta^2\mu-2\mu\gamma\right)+k^{2\alpha}\left(\pm 2\beta\lambda\sqrt{\beta^2+\gamma}-2\beta^2\lambda-2\lambda\gamma\right)\cdot$$

$$\cdot\left\{-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\}+$$

$$+k^{2\alpha}\left(\pm 2\beta\sqrt{\beta^2+\gamma}-2\beta^2-2\gamma\right)\cdot$$

$$\cdot\left\{-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\}^2 \tag{16}$$

where $\xi = kx + ct + \xi_0$.

When $\lambda^2 - 4\mu < 0$:

$$u_2(x,t)=-\frac{c^\alpha}{k^\alpha}\pm k^\alpha\lambda\sqrt{\beta^2+\gamma}\pm$$

$$\pm\ 2k^\alpha\sqrt{\beta^2+\gamma}\left\{-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\}$$

$$v_2(x,t)=k^{2\alpha}\left(\pm 2\beta\mu\sqrt{\beta^2+\gamma}-2\beta^2\mu-2\mu\gamma\right)+k^{2\alpha}\left(\pm 2\beta\lambda\sqrt{\beta^2+\gamma}-2\beta^2\lambda-2\lambda\gamma\right)\cdot$$

$$\cdot\left\{-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\}+$$

$$+\ k^{2\alpha}\left(\pm 2\beta\sqrt{\beta^2+\gamma}-2\beta^2-2\gamma\right)\cdot \tag{17}$$

$$\cdot\left\{-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}\xi^\alpha}\right]\right\}^2$$

where $\xi = kx + ct + \xi_0$.

Conclusions

In this paper, the existed fractional sub-equation method has been successfully obtained the exact solutions of the space-time fractional WBK equations. The above procedure shows that:

- the fractional sub-equation method is an efficient and powerful method in solving a wide class of equations, and
- the method is straightforward without any restrictive assumptions and special techniques. Whether we can introduce other new feasible algorithms to solve FDE, we hope this question will be further studied.

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