MODIFIED VARIATIONAL ITERATION METHOD FOR VARIANT BOUSSINESQ EQUATION

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In this paper, we solve the variant Boussinesq equation by the modified variational iteration method. The approximate solutions to the initial value problems of the variant Boussinesq equation are provided, and compared with the exact solutions. Numerical experiments show that the modified variational iteration method is efficient for solving the variant Boussinesq equation.

Key words: *modified variational iteration method, variant Boussinesq equation*

Introduction

Consider the variant Boussinesq equation:

$$u_t + (uv)_x + v_{xxx} = 0, \qquad v_t + u_x + vv_x = 0 \tag{1}$$

where u is the velocity, v – the total depth, and the subscripts denote partial derivatives [1-3]. As a model of water waves, eq. (1) describes the propagation of surface long wave towards two directions in a certain deep trough. The variant Boussinesq equation plays an important role in the fluid dynamics. Various wave solutions of eq. (1) have been studied by means of numerical or analytical methods, including homogeneous balance method [4], exp-function method [5], homotopy analysis method [6], extended Fan's sub-equation method [7], bifurcation theory [8], and algebraic method [9].

We are interested in the numerical simulation of the variant Boussinesq equation, which helps to realize the water waves. We will give the numerical solutions to the initial value problems associated with the variant Boussinesq equation by means of the modified variational iteration method (MVIM). The MVIM was proposed by Abassy, *et al.*, [10], for reducing the computation of repeated or unneeded terms in the original variational iteration method [11-14]. The approximate solutions by MVIM are compared with the exact solutions. Numerical experiments show the efficiency of the MVIM for the variant Boussinesq equation.

Modified variational iteration method

In order to illustrate the basic idea of MVIM, let us consider the partial differential equation:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad u(x,0) = f(x) \tag{2}$$

where $L = \partial/\partial t$, R is a linear operator with the partial derivative with respect to x, $Nu(x,t)$ – the non-linear term, and $g(x,t)$ – the inhomogeneous term.

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In the real applications, the cost of variational iteration method (VIM) may be high, since it requires the calculations of some repeated or unneeded terms. For speeding up the convergence and reducing the computation cost of VIM, the MVIM was proposed in [10]. The MVIM for eq. (2) is constructed by the variational iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t \{R[u_n(x,\xi)-u_{n-1}(x,\xi)]+[G_n(x,\xi)-G_{n-1}(x,\xi)]\}\,\mathrm{d}\xi$$

with $u_{-1}=0$, $u_0=f(x)$, $u_1=u_0-\int_0^t\{R(u_0-u_{-1})+(G_0-G_{-1})-g\}\mathrm{d}\xi$, and $G_n(x,t)$ is given by $Nu_n(x,t)=G_n(x,t)+O(t^{n+1})$.

Numerical experiments

In this section, two initial value problems of the variant Boussinesq equation are presented to illustrate the efficiency of the MVIM. We first consider the initial value problem of eq. (1) with the initial conditions:

$$u(x,0)=-c_1+c_1\tanh^2\left(\sqrt{-\frac{c_1}{2}}x\right) \text{ and } v(x,0)=-c-\sqrt{-2c_1}\tanh\left(\sqrt{-\frac{c_1}{2}}x\right)$$

As shown in [9], the exact soliton solutions to eq. (1) are given by:

$$u(x,t)=-c_1+c_1\tanh^2\left(\sqrt{-\frac{c_1}{2}}(x+ct)\right) \text{ and } v(x,t)=-c-\sqrt{-2c_1}\tanh\left(\sqrt{-\frac{c_1}{2}}(x+ct)\right)$$

respectively.

According to the MVIM, the iteration formulae can be written as:

$$\begin{aligned}u_{n+1}(x,t)&=u_n(x,t)-\int_0^t\{[v_{nxxx}(x,\xi)-v_{n-1xxx}(x,\xi)]+[G_n(x,\xi)-G_{n-1}(x,\xi)]\}\mathrm{d}\xi\\ v_{n+1}(x,t)&=v_n(x,t)-\int_0^t\{[u_{nx}(x,\xi)-u_{n-1x}(x,\xi)]+[H_n(x,\xi)-H_{n-1}(x,\xi)]\}\mathrm{d}\xi\end{aligned} \tag{3}$$

where $G_n(x,t)$ is obtained from $(u_nv_n)_x=G_n(x,t)+O(t^{n+1})$, and $H_n(x,t)$ is defined by $v_nv_{nx}=H_n(x,t)+O(t^{n+1})$. u_{-1}, v_{-1}, G_{-1}, and H_{-1} are set be zero.

By the iteration eq. (3) with $u_0=u(x,0)$ and $v_0=v(x,0)$, it follows the first order approximations:

$$u_1(x,t)=\frac{1}{16\cosh\left(\frac{1}{2}x\right)}\left(4+4\cosh x-\frac{2}{5}t\sinh x\right)$$

$$v_1(x,t)=-\frac{1}{10}-\tanh\left(\frac{1}{2}x\right)-\frac{1}{20}t\sec h^2\left(\frac{1}{2}x\right)$$

We remark that c =1/10 and c_1 = –1/2. The rest approximate solutions can be obtained similarly.

Tables 1 and 2 list the absolute errors of the MVIM solutions u_4 and v_4, respectively. Figure 1 plots the MVIM solution u_4 and the exact soliton solution $u(x, t)$ of eq. (1). The numerical results for the approximation v_4 and the soliton solution $v(x, t)$ are shown in fig. 2. We see that the MVIM performs well for this example. Precisely, the MVIM solutions agree well with the exact solutions.

We then consider the initial value problem of eq. (1) with the initial conditions:

$$u(x,0) = 2\,\mathrm{sec}\,h^2(x) \text{ and } v(x,0) = c + 2\,\tanh x$$

The solitary wave solutions to the above problem are given by [4]:

$$u(x,t) = 2\,\mathrm{sec}\,h^2(x-ct) \text{ and } v(x,t) = c + 2\,\tanh(x-ct)$$

respectively.

Table 1. Absolute errors of the MVIM solutions u_4

	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
1	$4.57\cdot10^{-8}$	$1.79\cdot10^{-8}$	$6.44\cdot10^{-9}$	$1.84\cdot10^{-10}$	$6.7\cdot10^{-10}$
2	$1.41\cdot10^{-6}$	$5.74\cdot10^{-7}$	$1.99\cdot10^{-7}$	$6.9\cdot10^{-9}$	$2.13\cdot10^{-8}$
3	$1.03\cdot10^{-5}$	$4.36\cdot10^{-6}$	$1.47\cdot10^{-6}$	$5.96\cdot10^{-8}$	$1.61\cdot10^{-7}$
4	$4.17\cdot10^{-5}$	$1.84\cdot10^{-5}$	$5.98\cdot10^{-6}$	$2.79\cdot10^{-7}$	$6.71\cdot10^{-7}$
5	$3.71\cdot10^{-5}$	$5.61\cdot10^{-5}$	$1.77\cdot10^{-5}$	$9.31\cdot10^{-7}$	$2.03\cdot10^{-6}$

Table 2. Absolute errors of the MVIM solutions v_4

	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
1	$9.27\cdot10^{-9}$	$1.42\cdot10^{-8}$	$7.23\cdot10^{-10}$	$1.49\cdot10^{-9}$	$8.82\cdot10^{-10}$
2	$3.2\cdot10^{-7}$	$4.44\cdot10^{-7}$	$1.99\cdot10^{-8}$	$4.76\cdot10^{-8}$	$2.79\cdot10^{-8}$
3	$2.59\cdot10^{-6}$	$3.3\cdot10^{-6}$	$1.28\cdot10^{-7}$	$3.6\cdot10^{-7}$	$2.09\cdot10^{-7}$
4	$1.16\cdot10^{-5}$	$1.36\cdot10^{-5}$	$4.45\cdot10^{-7}$	$1.51\cdot10^{-6}$	$8.69\cdot10^{-7}$
5	$3.71\cdot10^{-5}$	$4.05\cdot10^{-5}$	$1.09\cdot10^{-6}$	$4.6\cdot10^{-6}$	$2.62\cdot10^{-6}$

By the MVIM, we can construct the iteration eq. (3). If we set the initial approximations as $u_0 = 2\,\mathrm{sec}\,h^2(x)$ and $v_0 = c + 2\,\tanh x$ with $c = -1/20$, it follows the first order approximate solutions:

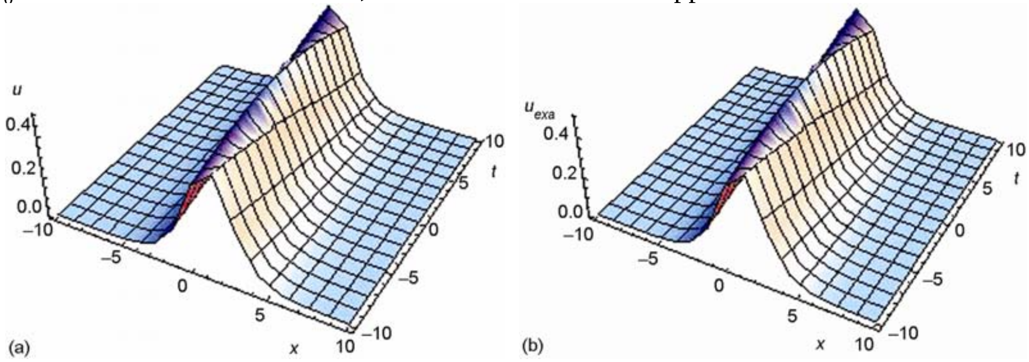

Figure 1. The approximation u_4 (a) and the soliton solution u (b) of eq. (1) when $-10 \le x, t \le 10$

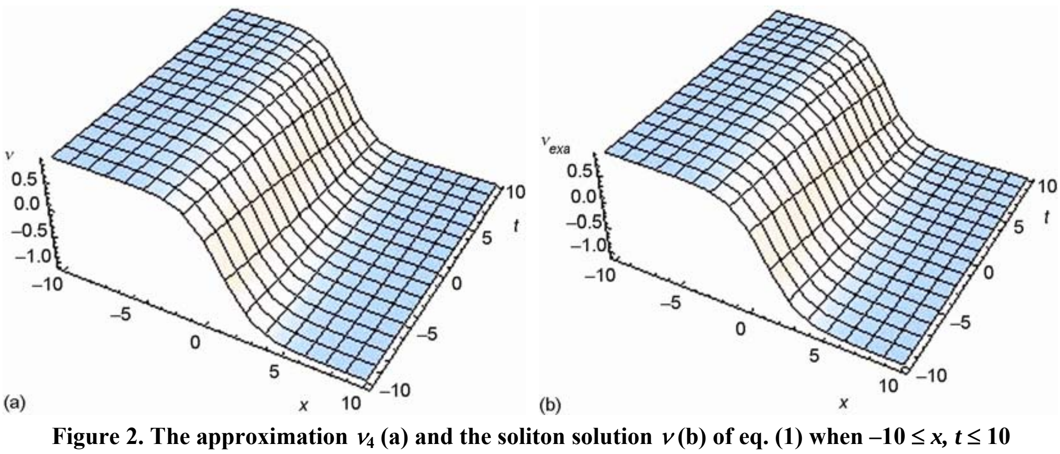

Figure 2. The approximation v_4 (a) and the soliton solution v (b) of eq. (1) when $-10 \le x, t \le 10$

$$u_1(x,t) = \sec h^2(x)\left(2 - \frac{1}{5}t\tanh(x)\right) \text{ and } v_1(x,t) = -\frac{1}{20} + 2\tanh x + \frac{1}{10}t\sec h^2(x)$$

As in the previous example, we can obtain the fourth order approximations u_4 and v_4 by applying eq. (3). For comparison, we plot the MVIM solution u_4 and the solitary wave solution $u(x, t)$ in fig. 3. Figure 4 shows the compared results for the approximation v_4 and the exact solution $v(x, t)$. The error curves for the approximations u_4 and v_4 are plotted in fig. 5. We also find that the MVIM works well for this initial value problem of the variant Boussinesq equation.

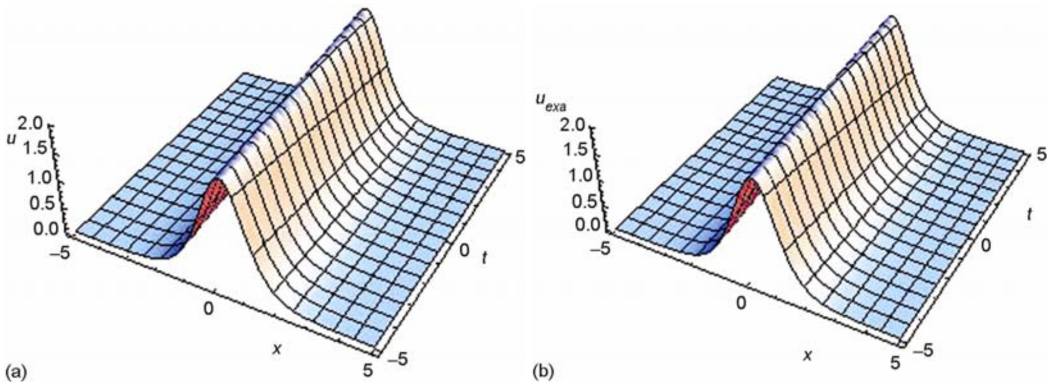

Figure 3. The approximation u_4 (a) and the exact solution u (b) when $-5 \leq x, t \leq 5$

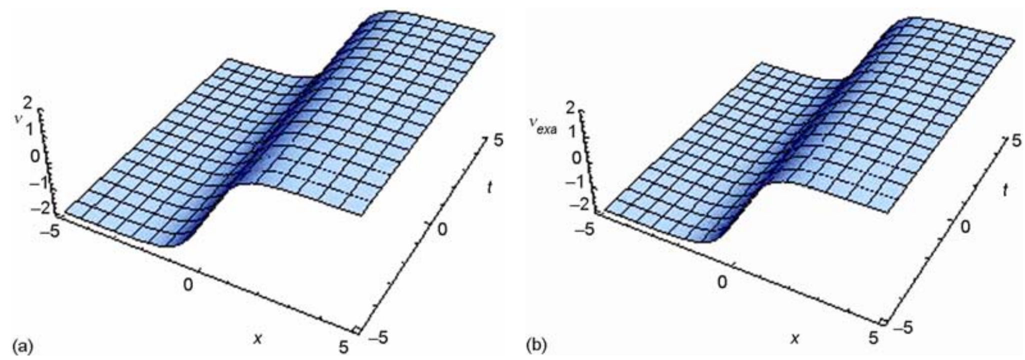

Figure 4. The approximation v_4 (a) and the exact solution v (b) when $-5 \leq x, t \leq 5$

Conclusions

This paper focuses on the variant Boussinesq equation by using the MVIM. The numerical results show the efficiency and advantage of this modified method. We will apply the MVIM to other non-linear partial differential equations in our future work.

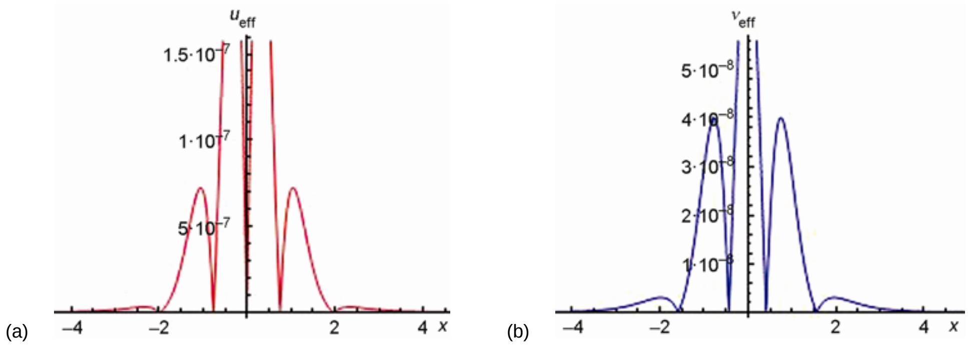

Figure 5. The error curves of u_4 (a) and v_4 (b) with $-5 \le x \le 5$ and $t = 1$

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