

DISCRETE FRACTIONAL DIFFUSION MODEL WITH TWO MEMORY TERMS

by

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Fractional calculus can always exactly describe anomalous diffusion. Recently the discrete fractional difference is becoming popular due to the depiction of non-linear evolution on discrete time domains. This paper proposes a diffusion model with two terms of discrete fractional order. The numerical simulation is given to reveal various diffusion behaviors.

Key words: *discrete fractional diffusion equation of two terms, discrete fractional difference*

Introduction

As is well known, the governing equations based on continuation assumption or differential models become invalid for porous media [1-3] which hold rich nanoscale pores. As a result, the fractal geometry and the fractional calculus are becoming the two effective tools. They have been considered into the application of permeability and heat diffusion [4-8].

Due to the beautiful non-locality of the fractional operators, the fractional calculus methods have been caught much attention. Particularly anomalous diffusion is superior to classical Darcy and Fourier laws. One of the most crucial models is the time fractional diffusion equation [9-11] and some efficient methods in both analytical and numerical methods were developed in the past ten years [12-20]. Some researchers pointed out that the fractal time or the discrete time can model anomalous diffusion or random walk. In view of this point, the tool of the discrete fractional calculus (DFC) developed by the Atici and Eloe [21, 22] was applied and discrete fractional modeling is suggested.

In this paper, we further considered a discrete fractional diffusion equation in two terms of fractional difference and the numerical simulation is given to discuss the diffusion behaviors.

Preliminaries

Definition 2.1 [18, 19] The fractional sum is defined as follows:

$${}_a\Delta_t^{-\nu}x(t) := \frac{h}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} [t - \sigma(sh)]_h^{(\nu-1)} x(a+sh), \quad t \in \{a + \nu h, a + \nu h + h, \dots\} \quad (1)$$

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where $t^{(\nu)}$ is the falling factorial function:

$$t^{(\nu)} = h^\nu \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \quad (2)$$

The numerical formulation of eq. (1) reads:

$${}_0\Delta_t^{-\nu} x(t) := \frac{h^\nu}{\Gamma(\nu)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1-j+\nu)}{\Gamma(n-j)} x(jh), \quad t = (n-1)h + \nu h \quad (3)$$

Definition 2.2 [23, 24]. For $0 < \nu \leq 1$, the Caputo difference on time scale is defined by:

$${}_0^C\Delta_t^\nu x(t) = \frac{h}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} [t-\sigma(sh)]_h^{(-\nu)} \Delta x(sh), \quad t \in \{(1-\nu)h, (1-\nu)h+h, \dots\} \quad (4)$$

which also can be re-written in a numerical form:

$${}_0^C\Delta_t^\nu u(t) = \frac{h^{1-\nu}}{\Gamma(1-\nu)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j-\nu)}{\Gamma(n-j)} \Delta u(jh) \quad (5)$$

for $t = (n-\nu)h$. Here $j = 1, 2, \dots, n-1$ means the first second, the second one, \dots , n^{th} second.

The DFC is a domain shifting operator. See for example in eq. (1), the domain of \mathbb{N}_0 is changed to \mathbb{N}_ν . Various domains of fractional sum and difference are summarized in Holm [14].

The difference order $\nu \neq 1$ leads to the memory effects and $[t-\sigma(s)]_h^{(\nu-1)}$ is the weight coefficients. In fact, from its equivalent form eq. (3), one can see that the fractional sum depends on the past status and one can vary the value of the dependences through the fractional order ν . The recent works are good examples to show this application in non-linear dynamics [25-27]. In view of this point, we consider the DFC for the anomalous diffusion.

Considering the complexity of diffusion through porous media and with the method in [28] which gave the discrete fractional diffusion equation as:

$${}_0^C\Delta_t^\nu u(x, t) = Ku_{xx}[x, t + (\nu-1)h] \quad (6)$$

we proposed the following discrete fractional diffusion of two terms.

Definition 2.3 For $0 < \beta < \alpha \leq 1$, u is the diffusion concentration, (x, t) is defined on $[0, L] \times [0, T]$, the discrete fractional diffusion of two terms is defined:

$${}_0^C\Delta_t^\alpha u[x, t + (1-\alpha)h] + \lambda {}_0^C\Delta_t^\beta u[x, t + (1-\beta)h] = Ku_{xx}(x, t), \quad t = 0, 1, 2, \dots \quad (7)$$

subjected to the initial boundary conditions:

$$u(x, 0) = \sin \frac{\pi x}{L}, \quad u(0, t) = u(L, t) = 0$$

In this model, λ is a constant and K is the diffusion coefficient. We introduce two more parameters and have more degrees of freedom of the fit.

Numerical simulation

Now let us give the numerical formulae. We can obtain the equations directly from eq. (5):

$$\frac{h^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j)} \Delta u(x, jh) + \frac{\lambda h^{1-\beta}}{\Gamma(1-\beta)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\beta)}{\Gamma(n-j)} \Delta u(x, jh) = K u_{xx}[x, (n-1)h] \quad (8)$$

The second order term can be numerically treated by the classical central difference:

$$u_{xx}(x, n-1) = \frac{u_{i+1}(n-1) - 2u_i(n-1) + u_{i-1}(n-1)}{\Delta x^2} \quad (9)$$

where $\Delta x = L/N$, $u_i(n-1) = u[i\Delta x, (n-1)h]$, $1 \leq i \leq N-1$.

As a result, we can derive a full discrete numerical formula:

$$\begin{aligned} \frac{h^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j)} \Delta u_i(jh) + \frac{\lambda h^{1-\beta}}{\Gamma(1-\beta)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\beta)}{\Gamma(n-j)} \Delta u_i(jh) = \\ = K \frac{u_{i+1}(n-1) - 2u_i(n-1) + u_{i-1}(n-1)}{\Delta x^2} \end{aligned} \quad (10)$$

Now let us discuss the diffusion behaviors through the numerical solutions with eq. (10). Let $N = 10$, $n = 200$, $K = 0.5$, $L = 1$, $\alpha = 0.8$, and $h = 0.05$. Fix the value of α and vary the second fractional difference order β . For different β , diffusion behaviors vs. discrete time are shown in fig. 1 and the 3-D numerical result is given in fig. 2.

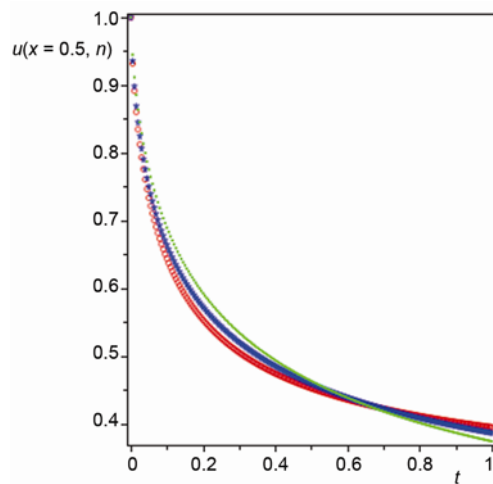


Figure 1. Diffusion behaviors vs. discrete time: the red “+”: $\alpha = 0.8$, and $\beta = 0.6$, the blue “*”: $\alpha = 0.8$ and $\beta = 0.4$, and the green “.”: $\alpha = 0.8$, $\beta = 0.4$

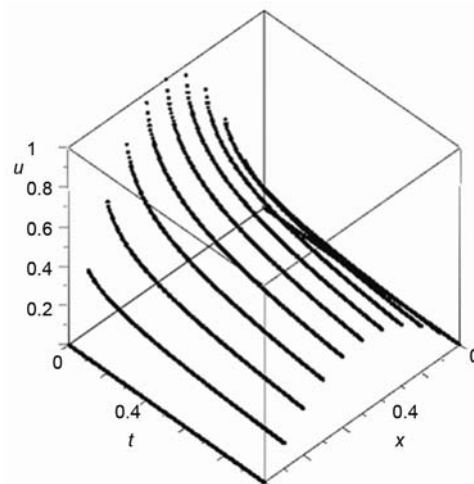


Figure 2. Diffusion behaviors vs. discrete time and continuous space; $\alpha = 0.8$, $\beta = 0.6$

Conclusions

In comparison with the continuous fractional models, the discrete time or media requires a novel tool for anomalous diffusion. This paper suggests the application of the DFC and a two term fractional difference diffusion equation is proposed. The numerical results show that the new model is effective for depicting the complexity of diffusion through porous media and becomes a potential tool for discrete fractional modeling.

Nomenclature

h	– size of time step, [s]
j	– integer, [–]
K	– diffusion coefficient, [m ² s ^{–1}]
L	– boundary of x , [m]
\mathbb{N}	– integer set, [–]
n	– integer, [–]
t	– time, [s]
T	– time, [s]
u	– concentration, [molcm ^{–3}]
x	– displacement, [m]

Greek symbols

α	– fractional difference order, [–]
β	– fractional difference order, [–]
Γ	– gamma function, [–]
${}_0^C\Delta_t^\nu$	– Caputo difference, [–]
$\Delta_0^{-\nu}$	– fractional sum, [–]
Δx	– size of space step, [m]
ν	– order of fractional sum, [–]

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