MODEL OF MOISTURE DIFFUSION IN FRACTAL MEDIA

by

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Moisture diffusion in fractal media does not obey the classical Fick's law. In this paper, its fractal partner is proposed to investigate the phenomenon in fractal media. It reveals that the moisture transport strongly depends on fractal dimensions of the media.

Key words: *moisture diffusion, fractal media, fractal derivative*

Introduction

Fractal media can be regarded as transport media with discontinuous geometry. Moisture diffusion in fractal media exists widely in natural and in many engineering fields, such as swelling of rise [1], insulation of transformer paper [2, 3], wood drying [4], moisture transport of fibers [5], comfort of textiles [6, 7], and so on. It has received much attention due to its significance in revealing the mechanism of mass transport in fractal media and guaranteeing the function and service life of products. Moisture transport in fractal media cannot be described by smooth continuum approach and needs the fractal nature of the objects to be taken into account because the transport performance in fractal media depends on the fractal dimensions [8-10]. In these cases, the classical Fick's law is not applicable, hence a new calculus must be developed to handle this fractal process. In this work, the fractal Fick's law is proposed by the fractal derivative proposed by He [11], and the allometric moisture diffusion problem is investigated.

Definition of fractal derivative

He [11] introduced a new fractal derivative from the geometric point of view for engineering application defined as:

$$\frac{Du(t)}{Dx^{\alpha}} = \lim_{\Delta x \to L_0} \frac{u(\mathrm{A}) - u(\mathrm{B})}{kL_0^{\alpha}} \tag{1}$$

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where k is a constant, α – the fractal dimension, L_0 – the smallest measure, and $u(t)$ – the distance between two points in the porous media.

An alternative definition is the fractional derivative defined as [12]:

$$\frac{DP}{Dx^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{t_0}^{t} (s-x)^{n-\alpha-1} [P_0(s) - P(s)]\, \mathrm{d}s \tag{2}$$

with $P_0(x)$ is a known function; it can be a solution of the problem when $\alpha = 1$.

Model for moisture diffusion in fractal media

In fractal media, moisture diffusion obeys fractal Fick's law expressed as:

$$\frac{\partial P}{\partial t} + \frac{D}{Dx^{\alpha}} \left(K \frac{DP}{Dx^{\alpha}} \right) = 0 \tag{3}$$

let

$$s = x^{\alpha} \tag{4}$$

Equation (3) is converted into an ordinary differential equation, which is:

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial s} \left(K \frac{\partial P}{\partial s} \right) = 0 \tag{5}$$

For a fractal media, the parameter s has a fractal physical meaning. If we assume the smallest measure is x, any discontinuity less than it is ignored, the distance between two points in a discontinuous space can be expressed as equation $s = kx^{\alpha}$, where k is a constant.

For a steady moisture diffusion, eq. (5) becomes:

$$K \frac{\partial P}{\partial s} = c \tag{6}$$

The solution is:

$$P = P_0 + \frac{c}{K} s \tag{7}$$

i. e.

$$P = P_0 + \frac{c}{K} x^{\alpha} \tag{8}$$

Equation (8) is valid for any adjacent points within the present scale.

Figure 1 shows the moisture diffusion path from point A to point B in a fractal media with different scales. In fig. 1(a), moisture diffusion path is measured with a measurement larger than scale L_0. In this case, the fractal feature of the media will be hindered, any discontinuity less than it is ignored. Thus, it is a continuous medium *i. e.* $\alpha = 1$. Equation (8) becomes a linear equation:

$$P = P_0 + \frac{c}{K} x \tag{9}$$

as illustrated in fig. 1(a).

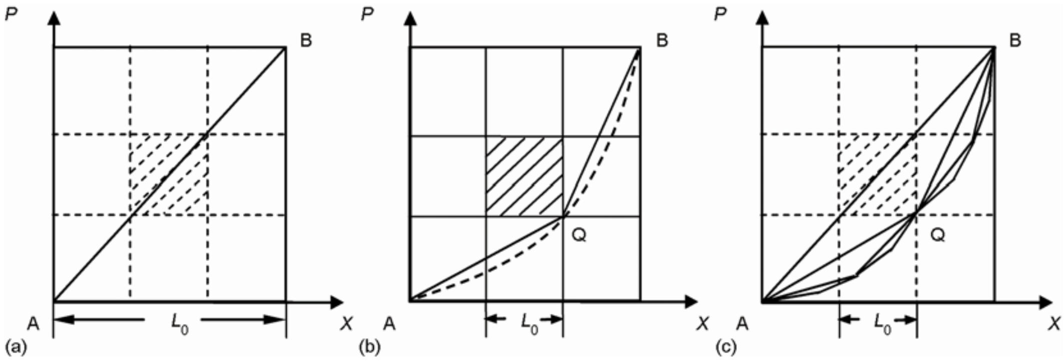

Figure 1. Moisture diffusion path in fractal media; (a) a larger scale, (b) a smaller scale, and (c) Koch curve model

When the scale tends to be smaller, *e. g.* 1/3 of that in fig. 1(a), see fig. 1(b), and any smaller scales will be ignored. In such case, diffusion through AQ and QB can be considered as a continuum medium, and the diffusion through AQB follows eq. (8) with fractal dimension $\alpha = \ln 8/\ln 3 = 1.893$, *i. e.*:

$$P = P_0 + \frac{c}{K} x^{1.893} \tag{10}$$

as the dotted line in fig. 1(b).

When the scale becomes smaller, AQ and QB has the self-similar patter as AQB, and Koch curve is obtained when iteration continues, see fig. 1(c). After each iteration, the number of sides of the Koch curve increases by a factor of 2, so the number of sides after n iterations is given by:

$$N_n = 2 \cdot 2^n \tag{11}$$

If the original equilateral triangle has sides (AQ and QB) of length *s*, the length of each side of the Koch curve after n iterations is:

$$S_n = \frac{s}{\left(\frac{3\sqrt{2}}{\sqrt{5}}\right)^n} \tag{12}$$

Therefore the length of the Koch curve after n iterations is:

$$L_n = N_n S_n = 2s \left(\sqrt{\frac{10}{9}}\right)^n \tag{13}$$

In each iteration, a new triangle is added on each side of the previous iteration, so the number of new triangles added in iteration n is:

$$T_n = N_{n-1} = 2 \cdot 2^{n-1} = 2^n \tag{14}$$

The area of each new triangle added in an iteration is $1/7.2$ of the area of each triangle added in the previous iteration, so the area of each triangle added in iteration n is:

$$a_n = \frac{a_{n-1}}{\frac{36}{5}} = \left(\frac{5}{36}\right)^n a_0 \tag{15}$$

where a_0 is the area of the original triangle AQB. The total new area added in iteration n is:

$$b_n = T_n\, a_n = 2^n \left(\frac{5}{36}\right)^n, \qquad a_0 = \left(\frac{5}{18}\right)^n a_0 \tag{16}$$

The total area enclosed by the dotted curve AQB and the straight line AB after n iterations is:

$$A_n = a_0 + \sum_{k=1}^{n} b_k = a_0 \left[1 + \sum_{k=1}^{n} \left(\frac{5}{18}\right)^k\right] \tag{17}$$

It is obvious that eq. (17) is continuous but non-differential anywhere. As the number of iterations tends to infinity, the limit of the length of the Koch curve is:

$$\lim_{n\to\infty} L_n = 2s\left(\sqrt{\frac{10}{9}}\right)^n \to \infty \tag{18}$$

The limit of the area is:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} a_0 \left[1 + \sum_{k=1}^{n} \left(\frac{5}{18}\right)^k\right] = a_0 \frac{1}{1-\frac{5}{18}} = \frac{18}{13} a_0 \tag{19}$$

Suppose the length of the scale L_0 in figs. 1(b) and 1(c) equals to 1, thus, the area of the original triangle AQB is:

$$a_0 = \frac{1}{2} 3\sqrt{2}\, \frac{\sqrt{2}}{2} = \frac{3}{2}$$

The area under the Koch curve in fig. 1(c) is:

$$\overline{A} = \frac{1}{2}\cdot 3^2 - \lim_{n\to\infty} A_n = \frac{9}{2} - \frac{18}{13}\cdot\frac{3}{2} = 2.423 \tag{20}$$

With boundary condition $P(0) = 0$, $P(3) = 3$, eq. (10) becomes:

$$P(x) = 0.375 x^{1.893} \tag{21}$$

The area under eq. (21) can be calculated as:

$$A = \int_0^3 0.375x^{1.893}\mathrm{d}x = 3.111 \tag{22}$$

The accuracy is 28%. Considering the practical applications, infinite iteration is forbidden. Though eq. (21) is less accurate, but the calculation is considerably shorter, and can be used for macroscale moisture diffusion in, for example, the woven and knitted fabric. When the yarn structure is considered, AQ and QB in fig. 1 can be modeled, respectively, as:

$$P = 0.269x^{1.893} \tag{23}$$

and

$$P = -0.730 + 0.466x^{1.893} \tag{24}$$

The area under the diffusion curve becomes:

$$A = \int_0^2 0.269x^{1.893}\mathrm{d}x + \int_2^3 (-0.730 + 0.466x^{1.893})\,\mathrm{d}x = 0.690 + 1.940 = 2.630 \tag{25}$$

The accuracy reaches 8.5%.

The scale can be even smaller, and the moisture diffusion between two adjacent points follows $P \propto x^{1.893}$, having typically self-similar patterns. When the scale tends to be zero, it becomes continuous but non-differential anywhere.

Conclusions

Moisture diffusion in the fractal media was investigated using the fractal derivative method. The result indicated that the process depends geometrically on the fractal media. In practical applications, it is forbidden for $L_0 \to 0$. For example, L_0 can be the distance between two warps or two wefts if we want to study the effect of warp-weft density on moisture diffusion, so that the fabric can be optimized. L_0 can also tend to nano/microscales if we want to take into account the influence of yarn's interior structure.

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