

## ON COMBINED EFFECT OF THERMAL RADIATION AND VISCIOUS DISSIPATION IN HYDROMAGNETIC MICROPOLAR FLUID FLOW BETWEEN TWO STRETCHABLE DISKS

by

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Original scientific paper  
<https://doi.org/10.2298/TSCI150325096A>

*In this paper, we investigate numerically the flow and heat transfer characteristics of a viscous incompressible electrically conducting micropolar fluid between two infinite uniformly stretching disks, taking the radiation and viscous dissipation effects into consideration. The transformed self similar coupled ordinary differential equations are solved using quasi-linearization method. The study may be beneficial in flow and thermal control of polymeric processing.*

Key words: *micropolar fluid, MHD, stretchable disks, radiatio, viscous dissipation, Nusselt number*

### Introduction

In the recent years, the investigation of flow over a stretching surface has attracted the attention of research community due to its significant applications in different industries such as extrusion paper production, extrusion of polymers sheet, metal and plastic industries [1-3]. The problem of fluid flow between parallel disks is also important due to its applications in many technological and engineering processes. These applications include semiconductor-manufacturing processes with rotating wafers, magnetic storage devices, gas turbine engines, hydro-dynamical machines and apparatus, crystal growth processes, rotating machinery, biomechanics, geothermal, geophysical, heat and mass exchanges, computer storage devices, viscometry, lubrication, oceanography radial diffusers, *etc.* Singh *et al.* [4] analyzed the inward flow between two stationery parallel disks and find out the solution by experimental as well as numerical methods. Fang *et al.* [5] determined exact solution of the Navier-Stokes equations analytically to study the MHD viscous flow under slip conditions over a permeable stretching surface. On the other hand, Volkan [6] used the analytic approach to find an approximate solution for the problem of flow between two disks rotating about distinct axes at different speeds. Ahmad *et al.* [7] used a similarity transformation to investigate the boundary layer flow of an electrically conducting fluid over a stretching plate. Yoon *et al.* [8] numerically investigated the flow and heat transfer near an infinite disk with surface roughness, which rotates steadily about the longitudinal axis. An analytical solution of axis-symmetric flow between two infinite stretching disks was presented by Robert *et al.* [9]. Fang and Zhang [10] gave the exact solution for the axis-symmetric flow between two stretchable infinite disks. Munawar *et al.* [11]

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employed the optimal homotopy analysis method to study the flow of an incompressible viscous fluid between two continuously stretching coaxial disks. On the other hand, flow of an electrically conducting fluid on a stretching rotating disk was studied by Turkyilmazoglu [12, 13] by means of similarity transformations. Attia [14] analyzed the problem of steady-flow of an incompressible viscous fluid over an infinite rotating disk through porous medium with heat transfer. Xinhui *et al.* [15] studied the asymmetric flow and heat transfer of viscous fluid between contracting/expanding rotating disks by using homotopy analysis method.

It has been noticed that the Newtonian model is not appropriate to completely describe some modern engineering and industrial processes which involve fluids, possessing an internal structure. Fluids having polymeric additives display a significant reduction of shear stress and polymeric concentration, as predicted experimentally by Hoyt and Fabula [16]. The deformation of such materials has been well explained by the theory of micropolar fluids introduced by Eringen [17]. The flow of colloidal solutions, liquid crystals, polymeric fluids, and blood are some of the examples where the micropolar fluid model may be employed. For further details regarding the applications of the micropolar fluid model, see [18, 19]. Micropolar fluid is an active area of research in which different aspects of the problems are being studied, in every possible detail. The problem of steady-axisymmetric flow and heat transfer in an incompressible micropolar fluid between two porous disks was studied by Takhar *et al.* [20] whereas the flow over an enclosed rotating disk was analyzed by Takhar *et al.* [21]. Similarly, MHD has attracted the research community due to its novel industrial applications. Excellent literature survey on the subject may be found in [22-24].

The previously mentioned researchers did not take the effects of viscous dissipation and thermal radiation in their investigations. Therefore, the aim of the present study is to investigate MHD steady viscous incompressible electrically conducting micropolar fluid flow and heat transfer between two stretching disks in the presence of a transverse magnetic field with radiation and viscous dissipation effects.

### Problem formulation

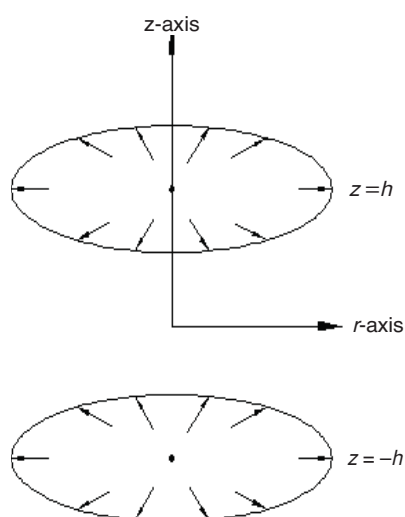


Figure 1. Physical configuration

We consider steady laminar viscous incompressible flow and heat transfer of an electrically conducting micropolar fluid between two stretchable infinite disks, located at  $z = -h$  and  $z = h$  as shown in fig. 1. A uniform transverse magnetic field  $\vec{B}$  is applied perpendicularly at the disks. The geometry of the problem suggests that the cylindrical polar co-ordinate system is most suitable for the study. Both the disks are stretching uniformly with the velocity proportional to the  $r$  co-ordinate. The magnetic Reynolds number is assumed to be small and hence the induced magnetic field can be neglected as compared to the imposed magnetic field [25]. We assume that there is no applied polarization voltage, so the electric field is zero.

Following, Hayat *et al.* [26], the governing momentum equations, in vector form, for the present problem are:

$$\nabla(\vec{V}) = 0 \quad (1)$$

$$(\mu + \kappa)\nabla^2\vec{V} + \kappa\nabla \times \vec{V} - \nabla p + \vec{J} \times \vec{B} = \rho\vec{V} \quad (2)$$

$$(\alpha + \beta + \gamma)\nabla(\nabla \vec{v}) - \gamma(\nabla \times \nabla \times \vec{v}) + \kappa \nabla \times \vec{V} - 2\kappa \vec{v} = \rho j \vec{v} \quad (3)$$

$$\vec{J} = \sigma_e (\vec{V} \times \vec{B}) \quad (4)$$

where  $\vec{V}$  is the velocity field,  $p$  – the pressure,  $\mu$  – the dynamic viscosity of the fluid,  $\kappa$  – the vortex viscosity,  $j$  – the micro-inertia per unit mass,  $\vec{v}$  – the microrotation vector,  $\rho$  – the fluid density,  $\mu$  and  $\kappa$  – the viscosity coefficients,  $\vec{J}$  – the current density,  $\vec{B}$  – the total magnetic field so that  $\vec{B} = \vec{B}_0 + \vec{b}$ ,  $\vec{b}$  – the induced magnetic field,  $\sigma_e$  – the electrical conductivity of the fluid, and  $\alpha$ ,  $\beta$  and  $\gamma$  are the gyro-viscosity coefficients. Furthermore  $\mu$ ,  $\kappa$ ,  $\alpha$ ,  $\beta$  and  $\gamma$  satisfy the following constraints:

$$2\mu + \kappa \geq 0, \quad \kappa \geq 0, \quad 3\alpha + \beta + \gamma \geq 0, \quad \gamma \geq |\beta|$$

The velocity and microrotation fields for the problem are:

$$\vec{V} = (u_r, 0, u_z) \text{ and } \vec{v} = (0, v_2, 0) \quad (5)$$

where,  $u_r = u_r(r, z)$ ,  $u_z = u_z(r, z)$ , and  $v_2 = v_2(r, z)$ .

These set of equations may be written in component form:

$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{h} \frac{\partial u_z}{\partial \eta} = 0 \quad (6)$$

$$\rho \left( u_r \frac{\partial u_r}{\partial r} + \frac{u_z}{h} \frac{\partial u_r}{\partial \eta} \right) = -\frac{\partial p}{\partial r} - \frac{\kappa}{h} \frac{\partial v_2}{\partial \eta} + (\mu + \kappa) \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{h^2} \frac{\partial^2 u_r}{\partial \eta^2} \right) - \sigma_e B_0^2 u_r \quad (7)$$

$$\rho \left( u_r \frac{\partial u_z}{\partial r} + \frac{u_z}{h} \frac{\partial u_z}{\partial \eta} \right) = -\frac{1}{h} \frac{\partial p}{\partial \eta} + \kappa \left( \frac{\partial v_2}{\partial r} + \frac{v_2}{r} \right) + (\mu + \kappa) \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{h^2} \frac{\partial^2 u_z}{\partial \eta^2} \right) \quad (8)$$

$$\rho j \left( u_r \frac{\partial v_2}{\partial r} + \frac{u_z}{h} \frac{\partial v_2}{\partial \eta} \right) = \kappa \left( \frac{1}{h} \frac{\partial u_r}{\partial \eta} - \frac{\partial u_z}{\partial r} \right) - 2\kappa v_2 + \gamma \left( \frac{\partial^2 v_2}{\partial r^2} - \frac{v_2}{r^2} + \frac{1}{r} \frac{\partial v_2}{\partial r} + \frac{1}{h^2} \frac{\partial^2 v_2}{\partial \eta^2} \right) \quad (9)$$

where  $\eta$  is the similarity variable and  $B_0$  – the strength of the magnetic field. Including the thermal radiation and viscous dissipation effects, the energy equation for the problem of flow between two stretching disks can be written:

$$\rho c_p \left( u_r \frac{\partial T}{\partial r} + \frac{u_z}{h} \frac{\partial T}{\partial \eta} \right) - k_0 \left( \frac{1}{h^2} \frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{1}{h} \frac{\partial q_r}{\partial \eta} - \frac{\mu}{h^2} \left( \frac{\partial u_r}{\partial \eta} \right)^2 = 0 \quad (10)$$

where  $T$  is the temperature,  $c_p$  – the specific heat capacity,  $k_0$  – the thermal conductivity of the fluid, and  $q_r$  – the radiative heat flux. In view of the Rosseland approximation for radiation [27], the radiative heat flux  $q_r$  is simplified:

$$q_r = -\frac{4\sigma}{3k^*} \frac{\partial T^4}{\partial z} \quad (11)$$

where  $\sigma$  and  $k^*$  are the Stefan-Boltzmann constant and the mean absorption coefficient, respectively. It is assumed that the temperature differences within the flow such that the term  $T^4$  may

be linearized. Hence, expanding  $T^4$  in a Taylor series about  $T_2$  and neglecting higher order terms, we get:

$$T^4 \cong 4T_2^3 T - 3T_2^4 \quad (12)$$

and therefore eq. (10) reduces to:

$$\rho c_p \left( u_r \frac{\partial T}{\partial r} + \frac{u_z}{h} \frac{\partial T}{\partial \eta} \right) - k_0 \left( \frac{1}{h^2} \frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{1}{h} \frac{\partial q_r}{\partial \eta} - \frac{\mu}{h^2} \left( \frac{\partial u_r}{\partial \eta} \right)^2 = 0 \quad (13)$$

The boundary conditions for the problem may be written as:

$$\left. \begin{aligned} u_r(r, -h) = rE, \quad u_r(r, h) = rE, \quad u_z(r, -h) = 0, \quad u_z(r, h) = 0 \\ v_2(r, -h) = 0, \quad v_2(r, h) = 0, \quad T(r, -h) = T_1, \quad T(r, h) = T_2 \end{aligned} \right\} \quad (14)$$

where  $E$  is the parameter determining the stretching strength of both the upper and lower disks, having units of  $1/t$ . The partial differential eqs. (7)-(9) and (10) can be converted into ordinary ones by using the following similarity transformations:

$$\eta = \frac{z}{h}, \quad u_r = -\frac{rE}{2} f'(\eta), \quad u_z = Eh f(\eta), \quad v_2 = -\frac{Er}{2h^2} g(\eta), \quad \theta(\eta) = \frac{T - T_2}{T_1 - T_2} \quad (15)$$

where  $T_1$  and  $T_2$  are the temperatures at the lower and upper disks, respectively. We see that the velocity field given in eq. (15) identically satisfies the continuity eq. (6) and hence represents a possible fluid motion. By using eq. (15) in eqs. (7)-(9) and (13), we get the following non-linear ordinary differential equations (ODE) in dimensionless form:

$$(1 + C_1) f'''' - C_1 g'' - R f f''' - R M^2 f'' = 0 \quad (16)$$

$$C_3 g'' + C_1 (f'' - 2g) + R C_2 \left( \frac{f'g}{2} - f g' \right) = 0 \quad (17)$$

$$\left( 1 + \frac{4}{3} Nr \right) \theta'' + \frac{1}{4} \text{Pr Ec} f''^2 - R \text{Pr} f \theta' = 0 \quad (18)$$

where  $R = \rho E h^2 / \mu$  is the stretching Reynolds number,  $M = (\sigma_e B_0^2 / \rho E)^{1/2}$  – the magnetic parameter,  $C_1 = \kappa / \mu$  – the vortex viscosity parameter,  $C_2 = j / h^2$  – the micro-inertia density parameter,  $C_3 = \gamma / \mu h^2$  – the spin gradient viscosity parameter,  $\text{Pr} = \mu c_p / k_0$  – the Prandtl number,  $Nr = 4\sigma T_2^3 / k^* k_0$  – the radiation parameter, and  $\text{Ec} = r^2 E^2 / c_p (T_1 - T_2)$  – the Eckert number. Boundary conditions given in eq. (10) also get the form:

$$f(-1) = f(1) = 0, \quad f'(-1) = -2, \quad f'(1) = -2, \quad g(-1) = 0, \quad g(1) = 0, \quad \theta(-1) = 1, \quad \theta(1) = 0 \quad (19)$$

### Computational procedure

We use quasi-linearization to construct the sequences of vectors  $\{f^{(k)}\}$ ,  $\{g^{(k)}\}$ , and  $\{\theta^{(k)}\}$ , which converge to the numerical solutions of eqs. (16)-(18), respectively. To construct  $\{f^{(k)}\}$  we linearize eq. (16), by retaining only the first order terms:

We set:

$$G(f, f', f'', f''', f''') \equiv (1 + C_1) f'''' - C_1 g'' - R f f''' - R M^2 f''$$

and

$$G[f^{(k)}, f'^{(k)}, f''^{(k)}, f'''^{(k)}, f^{(k+1)}] + [f^{(k+1)} - f^{(k)}] \frac{\partial G}{\partial f^{(k)}} + [f'^{(k+1)} - f'^{(k)}] \frac{\partial G}{\partial f'^{(k)}} + \\ + [f''^{(k+1)} - f''^{(k)}] \frac{\partial G}{\partial f''^{(k)}} + [f'''^{(k+1)} - f'''^{(k)}] \frac{\partial G}{\partial f'''^{(k)}} = 0$$

which simplifies to:

$$(1 + C_1)f^{(k+1)} - Rf'''^{(k+1)}f^{(k)} - RM^2f''^{(k+1)} - Rf^{(k+1)}f^{(k)} = C_1g^{(k)} - Rf^{(k)}f^{(k)} \quad (20)$$

Now eq. (20) gives a system of linear differential equations, with  $f^{(k)}$  being the numerical solution vector of the  $k^{th}$  equation. To solve the linear ODE, we replace the derivatives with their central difference approximations, giving rise to the sequence  $\{f^{(k)}\}$ , generated by the following linear system

$$Bf^{(k+1)} = C \text{ with } B \equiv B_{n \times n} [f^{(k)}] \text{ and } C \equiv C_{n \times 1} [f^{(k)}] \quad (21)$$

where  $n$  is the number of grid points. On the other hand, eqs. (17) and (18) are linear in  $g$  and  $\theta$ , respectively, and therefore, in order to generate the sequences  $\{g^{(k)}\}$  and  $\{\theta^{(k)}\}$ , we write:

$$C_3g^{(k+1)} + C_1[f''^{(k+1)} - 2g^{(k+1)}] + RC_2 \left[ \frac{f'^{(k+1)}g^{(k+1)}}{2} - g'^{(k+1)}f^{(k+1)} \right] = 0 \quad (22)$$

$$\left( 1 + \frac{4}{3}Nr \right) \theta''^{(k+1)} + \frac{1}{4} \text{PrEc} f''^{(k+1)2} - R \text{Pr} f'^{(k+1)} \theta'^{(k+1)} = 0 \quad (23)$$

Importantly  $f^{(k+1)}$  is considered to be known in the previous equation and its derivatives are approximated by the central differences.

We outline the computational procedure as follows.

- Provide the initial guess  $f^{(0)}$ ,  $g^{(0)}$ , and  $\theta^{(0)}$ , satisfying the boundary conditions given in eq. (19).
- Solve the linear system given by eq. (21) to find  $f^{(1)}$ .
- Use  $f^{(1)}$  to solve the linear system arising from the finite difference discretization of eqs. (22) and (23), to get  $g^{(1)}$  and  $\theta^{(1)}$ .
- Take  $f^{(1)}$ ,  $g^{(1)}$ , and  $\theta^{(1)}$  as the new initial guesses and repeat the procedure to generate the sequences  $\{f^{(k)}\}$ ,  $\{g^{(k)}\}$ , and  $\{\theta^{(k)}\}$  which, respectively, converge to  $f$ ,  $g$ , and  $\theta$  (the numerical solutions of eqs. (16)-(18)).

The three sequences are generated until:

$$\max \left\{ \|f^{(k+1)} - f^{(k)}\|_{L_\infty}, \|g^{(k+1)} - g^{(k)}\|_{L_\infty}, \|\theta^{(k+1)} - \theta^{(k)}\|_{L_\infty} \right\} < 10^{-6}$$

It is important to note that the coefficient matrix  $B$  in eq. (21) will be penta-diagonal and not diagonally dominant, and hence the iterative method (like SOR) may fail or work very poorly. Therefore, some direct method like LU factorization or Gaussian elimination with full pivoting (to ensure stability) may be employed. On the other hand, eqs. (22) and (23) will give rise to the diagonally dominant algebraic system when discretized using the central differences, which allows us to use the SOR method. Lastly, we may also improve the order of accuracy of the solution by using polynomial extrapolation scheme.

## Results and discussion

In this section we present our findings in tabular and graphical forms together with the discussion and their interpretations. Our objective is to develop a better understanding of the effects of micropolar structure of fluids on flow and heat transfer characteristics. The parameters of the study are the magnetic parameter,  $M$ , the Reynolds number,  $Re$ , the micropolar parameters,  $C_1$ ,  $C_2$  and  $C_3$ , the Eckert number,  $Ec$ , the radiation parameter,  $Nr$ , and the Prandtl number,  $Pr$ .

Following Hayat and Nawaz [19] and Hayat *et al.* [26], the skin friction  $C_f$ , the wall couple stress  $C_g$ , and the Nusselt number at  $z = h$  are:

$$C_f = \frac{\tau_w}{\rho(u_w)^2} = \frac{(\mu + \kappa)}{\rho(rE)^2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \bigg|_{z=h} = -\frac{1}{2} \frac{(1 + C_1)}{R} f''(1)$$

$$C_g = \frac{\gamma \frac{\partial v_2}{\partial z} \big|_{z=h}}{\rho(u_w)^2} = \frac{\gamma \frac{\partial v_2}{\partial z} \big|_{z=h}}{\rho(rE)^2} = \frac{1}{2} \frac{C_3}{R} g'(1)$$

and

$$Nu = \frac{h q_w}{k(T_1 - T_2)} = -\frac{hk \frac{\partial T}{\partial z} \big|_{z=h}}{k(T_1 - T_2)} = -\theta'(1)$$

where  $R = \rho E h r / \mu$  is the local Reynolds number.

**Table 1. Five cases of values of micropolar parameters  $C_1$ ,  $C_2$  and  $C_3$**

Case No.	$C_1$	$C_2$	$C_3$
1 (Newtonian)	0	0	0
2	2	0.2	0.3
3	4	0.4	0.5
4	6	0.6	0.7
5	8	0.8	0.9

We will study the effects of the parameters described above on  $C_f R$ ,  $C_g R$ , and  $Nu$ , as well as on the velocity profiles  $f(\eta)$ ,  $f'(\eta)$ , the microrotation profile  $g(\eta)$  and the temperature profile  $\theta(\eta)$ .

All the cases of the micropolar parameters in the present work are shown in tab. 1. In order to establish the validity of our numerical computations and to improve the order of accuracy of the solutions, numerical results are computed for three grid sizes  $h$ ,  $h/2$  and  $h/4$  and then Richardson extrapolation is used as presented in tab. 2. It also shows the convergence of our numerical results as the step size decreases.

Table 3 predicts that both the skin friction coefficient,  $C_f Re_r$ , and the couple stress coefficient,  $C_g$

$Re_r$ , as well as the Nusselt number increase as the stretching Reynolds number,  $R$ , increases. The increased stretching rate of the disks forces the fluid to move rapidly towards the disks, thus increasing both the shear and couple stresses. Moreover, the fluid is carrying away the heat from the flow region, resulting in increasing the temperature difference and hence the heat transfer rate.

Table 4 shows that the magnetic parameter increases both the skin friction and the couple stress coefficient while reducing the Nusselt number at the disks. From the mechanical point of view, the magnetic field exerts a friction like force, called the Lorentz force, which tends to drag the fluid towards the disks.

**Table 2. Dimensionless temperature  $\theta(\eta)$  on three grid sizes and extrapolated values for  $R = 50$ ,  $M = 1$ ,  $C_1 = 2$ ,  $C_2 = 0.4$ ,  $C_3 = 0.2$ ,  $Pr = 0.5$ ,  $Ec = 0.2$**

$\eta$	1 <sup>st</sup> grid	2 <sup>nd</sup> grid	3 <sup>rd</sup> grid	Extrapol. values
-0.6	0.611051	0.611063	0.611066	0.611067
-0.2	0.519907	0.519904	0.519904	0.519904
0	0.500000	0.500000	0.500000	0.500000
0.2	0.480093	0.480095	0.480096	0.480096
0.6	0.388949	0.388936	0.388933	0.388932

**Table 3. The effect of stretching Reynolds number on  $Re_r C_f$ ,  $Re_r C_g$  and  $Nu$  for  $M = 1$ ,  $C_1 = 2$ ,  $C_2 = 0.2$ ,  $C_3 = 0.3$ ,  $Pr = 0.7$ ,  $Nr = 1$ ,  $Ec = 0.2$**

$R$	$Re_r C_f$	$Re_r C_g$	$Nu$
0	7.7391	1.1584	0.6812
10	10.9395	1.2740	0.9039
20	13.7080	1.3347	1.0969
30	16.0791	1.3674	1.2691
40	18.1509	1.3852	1.4266

This not only results in increasing the skin friction coefficient at the disks but also causes greater spinning of the micro-fluid particles, and hence increases the couple stress coefficient as well. Furthermore, the frictional force tends to raise the fluid temperature and thus decreases the temperature difference between the fluid and the disks. Therefore, the heat transfer rate, which is directly proportional to the temperature difference, also decreases. The influence of the micropolar parameters  $C_1$ ,  $C_2$ , and  $C_3$  on the shear and the couple stresses is given in tab. 5. The first case corresponds to the Newtonian fluid whereas the remaining ones are taken arbitrarily to investigate their influence on the flow as chosen in the literature [28-31]. It may be concluded that the micropolar structure of the fluid tends to increase the skin friction coefficient while causing the microrotation in the fluid which is responsible for the increase in the couple stress coefficient at the disks, as shown in the tab. 5. It is also clear from the table that the role of the micro-fluid particles in increasing the heat transfer rate is not as pronounced as compared to its effect on the shear and couple stresses. It is due to the reason that micropolar parameters do not appear in the heat eq. (14) and therefore do not directly influence the heat transfer characteristics of the problem.

**Table 4. The effect of magnetic parameter on  $Re_r C_f$ ,  $Re_r C_g$ ,  $Nu$  with  $R = 20$ ,  $C_1 = 2$ ,  $C_2 = 0.2$ ,  $C_3 = 0.3$ ,  $Pr = 0.7$ ,  $Nr = 1$ ,  $Ec = 0.2$**

$M$	$Re_r C_f$	$Re_r C_g$	$Nu$
0	10.5522	1.2595	1.1645
0.5	11.4483	1.2816	1.1413
1.0	13.7080	1.3347	1.0969
1.5	16.6320	1.3970	1.0616
2.0	19.8207	1.4560	1.0419

**Table 5. The effect of micropolar parameters on  $Re_r C_f$ ,  $Re_r C_g$ ,  $Nu$  with  $R = 20$ ,  $M = 1$ ,  $Pr = 0.7$ ,  $Nr = 1$ ,  $Ec = 0.2$**

Cases	$Re_r C_f$	$Re_r C_g$	$Nu$
1	7.3418	0	1.0369
2	13.7080	1.3347	1.0969
3	18.9298	2.4190	1.1429
4	23.8937	3.4572	1.1732
5	28.7612	4.4781	1.1944

An increase in the Prandtl number always results in increasing the Nusselt number at the disks as shown in tab. 6. An increase in Prandtl number means that the heat energy is taken away from the flow region so that the temperature difference between the fluid and the disks increases, which results in enhancing the heat transfer rate at the disks. It is clear from tab. 7 that the effect of the radiation number is to decrease the heat transfer rate at the disks whether we consider or ignore the effects of the viscous dissipation. Table 8 shows that the viscous dissipation may cause thermal reversal at the lower disk while increasing the Nusselt number at the upper one, thus decreasing the temperature of the fluid which in turn increases temperature difference between the fluid and the upper disk, and hence the heat transfer rate at the upper disk.

**Table 6. The effect of Prandtl number on Nu with  $R = 3$ ,  $M = 1$ ,  $C_1 = 2$ ,  $C_2 = 0.2$ ,  $C_3 = 0.3$ ,  $Nr = 2$** 

Pr	Ec = 0.0	Ec = 0
	Nu	Nu
3	0.6777	1.5331
8	1.0189	4.1180
13	1.3757	8.8125
18	1.7133	18.3785
21	1.9004	29.0152

**Table 7. The effect of radiation parameter on Nu with  $R = 20$ ,  $M = 1$ ,  $C_1 = 2$ ,  $C_2 = 0.2$ ,  $C_3 = 0.3$ ,  $Pr = 0.7$** 

Nr	Ec = 0.3	Ec = 0.3
	Nu	Nu
0.0	1.4399	2.1557
0.1	1.3196	1.9109
0.5	1.0319	1.3822
3.0	0.6576	0.7575
10.0	0.5522	0.5856

**Table 8. The effect of viscous dissipation on  $\theta'(-1)$  Nu with  $R = 20$ ,  $M = 1$ ,  $C_1 = 2$ ,  $C_2 = 0.2$ ,  $C_3 = 0.3$ ,  $Pr = 0.7$ ,  $Nr = 1.0$** 

Ec	$-\theta'(-1)$	Nu
0.0	0.864027	0.864027
0.3	0.514688	1.213367
0.6	0.165348	1.562707
0.9	-0.183992	1.912046
1.2	-0.533331	2.261386

For eq. (16), the error residual after  $k^{th}$  is denoted by  $Re(f^{(k)})$  and is defined:

$$Re[f^{(k)}] = (1 + C_1)f^{(k)''''} - C_1g^{(k)''} - Rf^{(k)}f^{(k)''} - RM^2f^{(k)''}$$

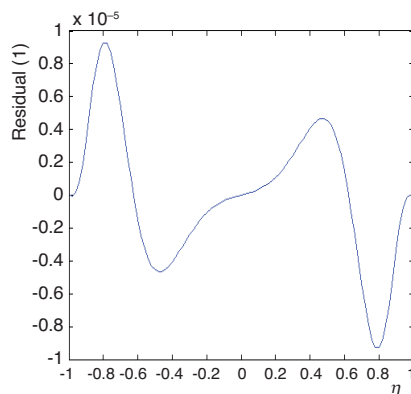
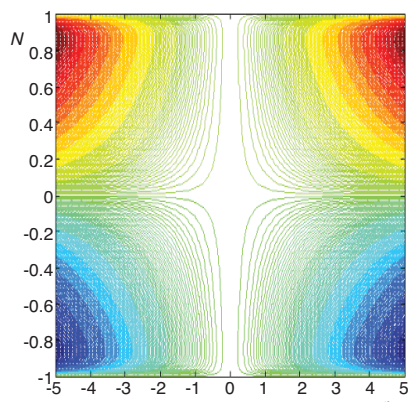
**Figure 2. Error residual of  $f(\eta)$  after three iterations****Figure 3. Streamlines for the problem**

Figure 2 shows  $Re(f^{(k)})$  after three iterations of the previously mentioned computational procedure on a grid with 101 points. It is interesting to see that the maximum value of the error residual has dropped to  $10^{-5}$  within just three iterations, which reflects the efficiency of the numerical algorithm.

Now we present the graphical interpretation of our results. Streamlines for the present problem are given in fig. 3. It is obvious that the streamlines near the walls are very close to each other showing larger gradients of the stream function which, in turn, predicts greater fluid velocity closer to the disks. Figures 4-8 predict the influence of the stretching Reynolds number for a typical value of the magnetic parameter, the micropolar parameters, the Eckert number, the Prandtl number, and the radiation parameter. The stretching Reynolds number decreases the velocity as well as the microrotation distribution across the disks, figs. 4-6. Whether we consider the viscous dissipation effects or not, Reynolds number always tends to flatten the temperature profiles almost in the middle of the two disks, thus developing an equivalent temperature region. On the other hand, it discourages the thermal reversal near the lower disk, for the case  $Ec \neq 0$ . From figs. 9-12, it is clear that the effect of  $M$  on the velocity and microrotation distribution is similar to that of  $R$ . On the other hand, the external magnetic field decreases the thermal reversal by decreasing the temperature distribution across the disks.



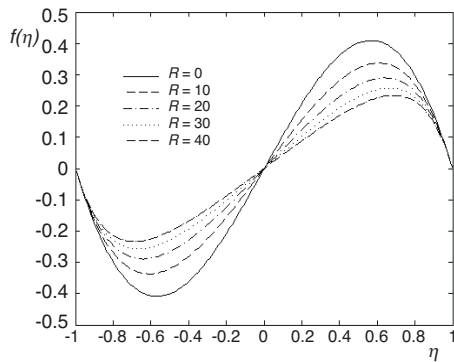


Figure 4. Variation of axial velocity for various  $R$

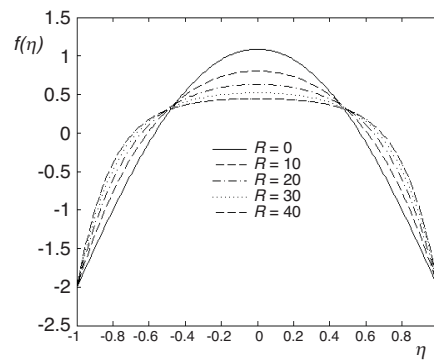


Figure 5. Variation of radial velocity for various  $R$

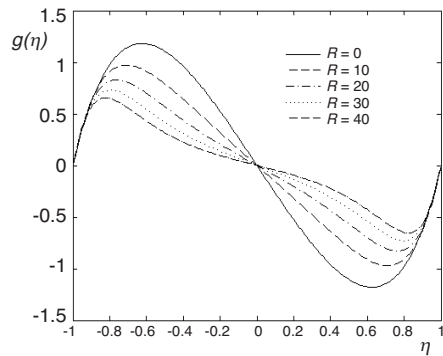


Figure 6. Variation of microrotation for various  $R$

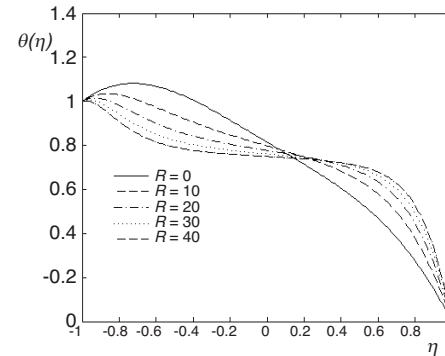


Figure 7. Variation of temperature for various  $R$

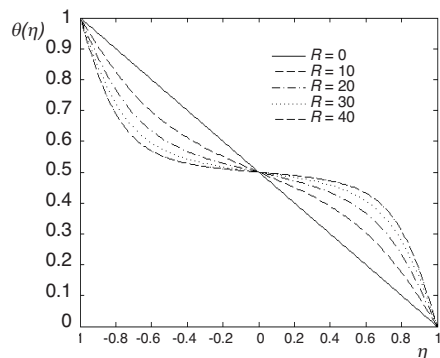


Figure 8. Variation of temperature for various  $R$

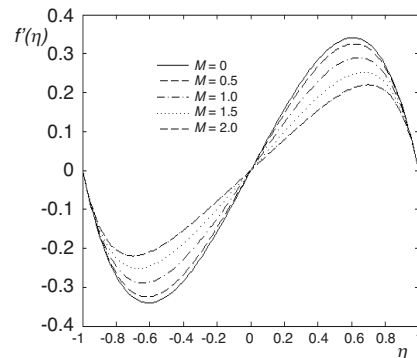
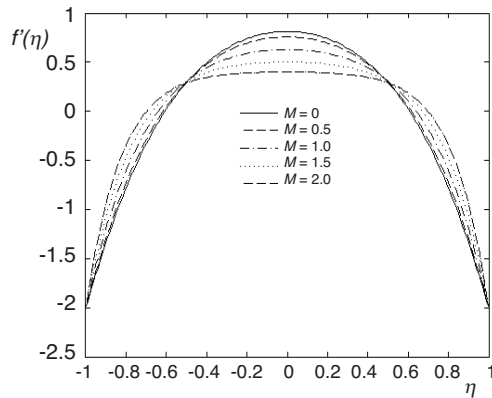
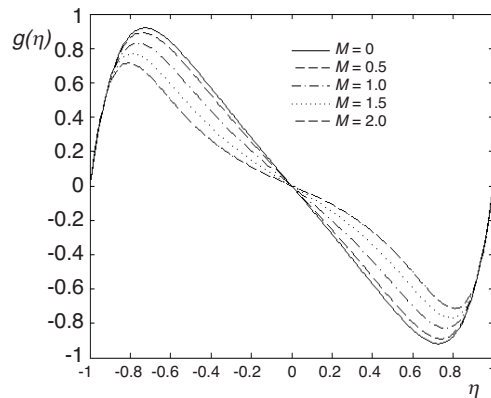
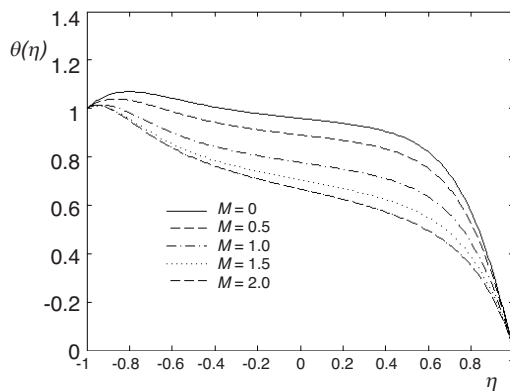
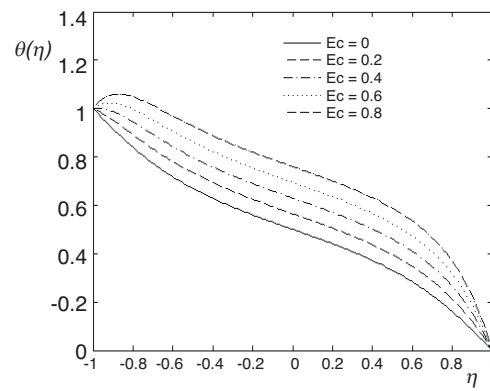
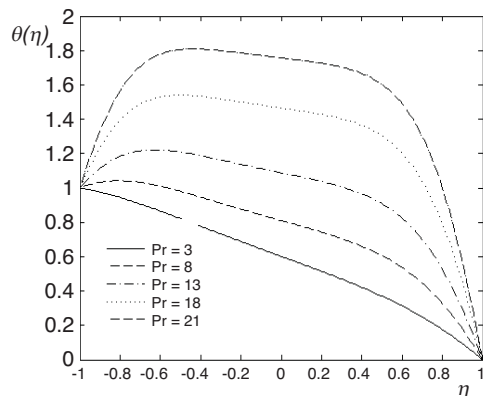
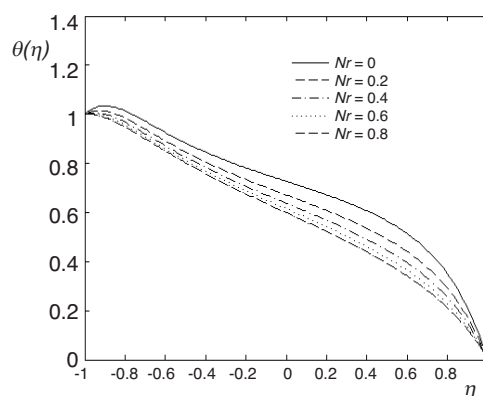


Figure 9. Axial velocity for various  $M$

It is noted that the effect of the micropolar structure of the fluid on the velocity, microrotation and temperature profiles is opposite to that of the magnetic field. Thus, the external magnetic field tends to balance the effect of the micropolar parameters. Figure 13 shows that the viscous dissipation tends to eliminate the symmetry of the temperature profiles by raising them near the lower disks, thus causing the thermal reversal. Finally, the effect of the Prandtl number on the heat profiles is opposite to that of radiation parameter, figs. 14 and 15, both in the presence and absence of the viscous dissipation.

Figure 10. Radial velocity for various  $M$ Figure 11. Microrotation for various  $M$ Figure 12. Temperature for various  $M$ Figure 13. Temperature for various  $Ec$ Figure 14. Variation of temperature  
for various  $Pr$ Figure 15. Variation of temperature  
for various  $Nr$ 

## Conclusions

In this paper, we numerically study how the governing parameters affect the flow and heat transfer characteristics of the steady-laminar, incompressible of an electrically conducting micropolar fluid between two stretchable infinite disks. Following conclusions have been drawn.

Micropolar fluids exhibit significant rise in the skin friction coefficient at the disks compared to the Newtonian ones, which may be beneficial for many industrial processes (e. g., in flow and thermal control of polymeric processing). Viscous dissipation may cause thermal reversal near the lower disk while significantly raising the temperature profile, whereas the external magnetic field magnifies the thermal reversal but lowers the temperature distribution, away from the lower disk. The external magnetic field, in this way, supports the viscous dissipation near the lower disk but opposes it away from the disk. Therefore, the combined effect of the external magnetic field and viscous dissipation may be taken care of while simulating the flow between the disks. The effect of the radiation number is to decrease the Nusselt number rate at the disks.

### Nomenclature

$B_0$  – strength of magnetic field, [ $\text{kgs}^{-2}\text{A}$ ]  
 $C_f$  – skin friction, [–]  
 $C_g$  – wall couple stress, [–]  
 $C_1$  – vortex viscosity, ( $=\kappa/\mu$ ), [–]  
 $C_2$  – microinertia density parameter, ( $=j/h^2$ ), [–]  
 $C_3$  – spin gradient viscosity parameter, ( $=\gamma/\mu h^2$ ), [–]  
 $c_p$  – specific heat capacity, [ $\text{Jkg}^{-1}\text{K}^{-1}$ ]  
 $E$  – stretching strengths of the lower and upper disks, respectively, [ $\text{Nt}^{-1}$ ]  
 $\text{Ec}$  – Eckert number, [ $=r^2 E_1^2 / c_p (T_1 - T_2)$ ], [–]  
 $f, f'$  – dimensionless axial and radial velocities, [–]  
 $g$  – dimensionless microrotation, [–]  
 $h$  – distance between the disks, [m]  
 $k_0$  – thermal conductivity of the fluid, [ $\text{Wm}^{-1}\text{K}^{-1}$ ]  
 $k^*$  – mean absorption coefficient, [–]  
 $M$  – magnetic parameter, ( $=\sigma_e B^2 / \rho E_1$ )<sup>1/2</sup>, [–]  
 $Nr$  – radiation parameter, ( $=4\sigma T_\infty^3 / k^* k_0$ ), [–]  
 $\text{Nu}$  – Nusselt number, [–]  
 $n$  – number of grid points, [–]  
 $\text{Pr}$  – Prandtl number, ( $=\mu c_p / k_0$ ), [–]  
 $p$  – pressure, [ $\text{Nm}^{-2}$ ]  
 $q_r$  – radiative heat flux, [ $\text{Wm}^{-2}$ ]  
 $q_w$  – heat transfer rate at the disks, [ $\text{Wm}^{-2}$ ]  
 $R$  – stretching Reynolds number, ( $=\rho E_1 h^2 / \mu$ ), [–]  
 $T$  – temperature, [K]

$u, v, w$  – velocity components along transverse, radial and axial component of velocities, respectively, [ $\text{ms}^{-1}$ ]  
 $u_r, u_z$  – radial and axial components of velocity, respectively, [ $\text{ms}^{-1}$ ]  
 $u_w$  – stretching velocity of the disks, [ $\text{ms}^{-1}$ ]

### Greek symbols

$\alpha, \beta, \gamma$  – gyroviscosity coefficients, [–]  
 $\eta$  – similarity variable, [–]  
 $\theta$  – dimensionless temperature, [–]  
 $\mu$  – dynamic viscosity, [ $\text{Nsm}^{-2}$ ]  
 $\rho$  – density, [ $\text{kgm}^{-3}$ ]  
 $\sigma$  – Stefan-Boltzmann constant, [ $\text{Wm}^{-2}\text{K}^{-4}$ ]  
 $\sigma_e$  – electrical conductivity, [ $\text{Sm}^{-1}$ ]  
 $\tau_w$  – shear stress at the disks, [ $\text{Nm}^{-2}$ ]  
 $v_1, v_2, v_3$  – component of microinertia along the transverse, radial and axial direction, respectively, [ $\text{kgm}^2$ ]

### Subscripts

1 – condition at the lower disk  
 2 – condition at the upper disk

### Superscripts

' – differentiation w. r. t.  $\eta$   
 $\overline{(\cdot)}$  – term number in the sequence

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