UNSTEADY MAGNETOHYDRODYNAMICS THIN FILM FLOW OF A THIRD GRADE FLUID OVER AN OSCILLATING INCLINED BELT EMBEDDED IN A POROUS MEDIUM

by

Fazal GHANI[a], Taza GUL[a], Saeed ISLAM[a], R. A. SHAH[b], Ilyas KHAN[c], Shafie SHARIDA[c*]. S. NASIR[a], and M. A. KHAN[a]

[a] Department of Mathematics, Abdul Wali King University Mardan, Khyber Pakhtunkhwa, Pakistan
[b] Department of Mathematics, UET Peshawar, Khyber Pakhtunkhwa, Pakistan
[c] Department of Mathematical Sciences, Faculty of Science, University Teknology Malaysia, UTM Johor, Bahru, Johor, Malaysia

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In the present work we examine the motion of an incompressible unidirectional magnetohydrodynamics thin film flow of a third grade fluid over an oscillating inclined belt embedded in a porous medium. Moreover, heat transfer analysis has been also discussed in the present work. This physical problem is modeled in terms of non-linear partial differential equations. These equations together with physical boundary conditions are solved using two analytical techniques namely optimal homotopy asymptotic method and homotopy perturbation method. The comparisons of these two methods for different time level are analyzed numerically and graphically. The results exposed that both methods are in closed agreement and they have identical solutions. The effects of various non-dimensional parameters have also been studied graphically.

Key words: *unsteady thin film flows, magnetohydrodynamics, porous medium, third grade fluid, heat transfer, inclined belt, optimal homotopy asymptotic method, homotopy analysis method*

Introduction

Non-Newtonian fluid flow and heat transfer acting a vital position in numerous technological and industrials manufacturing processes. A number of applications of non-Newtonian flow are found in drilling mud, polymer solution, drilling of gas and oil wells, glass fiber, and paper production. Third grade fluid is one of the significant sub-classes of non-Newtonian fluid. In literature the study of third grade fluid flow through various geometrical planes has received enormous concentration from scientists and engineers. Gul *et al*. [1] investigated unsteady second grade thin film fluid on a vertical belt. They studied thin film fluid motion at different time level. Aiyesimi *et al*. [2, 3] studied the cause of slip boundary on MHD third order fluid through inclined plane in the presence of heat transfer. They solved the problem by using regular and homotopy perturbation methods (HPM) and discussed the effect of physical parameters graphically. Abdullah [4] used HAM for the solution of nonlinear problems. Gul *et al*. [5, 6] studied thin film flow of third order fluid in lifting and drainage problems for constant and variable viscosities. For solutions they used two analytical methods adomian decomposition method (ADM) and optimal homotopy asymptotic method (OHAM) for lifting and drainage velocity and temperature profiles. In their work, they also

*Corresponding author, e-mail: sharidan@utm.my

examined the comparison of these two solutions graphically and numerically. Some other interesting non-linear problems related to the present work are studied in [7]. Hayat *et al.* [8] discussed second grade fluid with convection boundary conditions. They used transformation to reduce partial differential equations to ordinary differential equations.

The flow through porous medium has countless applications in science and engineering such as petroleum, chemical engineering, chemical reactor, irrigation, and drainage. Hayat *et al.* [9] discussed the unsteady flow of third grade in a porous space. In the modeling of fluid flow they used the modified Darcy's law. Gamal [10] studied the thin film flow of unsteady micro polar fluid through porous medium in the presence of MHD. They have been used numerical techniques to solve the problem. In their work the effects of the various modeled parameters have been presented graphically. Sajid *et al.* [11] discussed the thin film flow of fourth order fluid through a vertical cylinder.

The main objective if this works to study the unsteady MHD thin film of non--Newtonian fluid with heat transfer on an inclined oscillating belt using OHAM and HPM. Idrees *et al.* [12] discussed the axisymmetric flow of incompressible fluid between two parallel plats and analytic solutions are obtained using OHAM, HPM, and perturbation method. It was shown that OHAM solutions are more precise and accurate. Siddiqui *et al.* [13] studied the thin film flow of non-Newtonian fluid on inclined plane. The problem has been solved for velocity filed by using OHAM and perturbation technique. Mabood *et al.* [14] discussed the OHAM method for non-linear Riccati differential equation. Ganji and Rafei [15] investigated the HPM method for the solution of Hirota Statsuma coupled partial differential equations. Lin [16] studied the solution of partial differential equation using HPM. Nawaz *et al.* [17] studied the approximate solution of Burger's equations using OHAM and compared the solution with ADM. Hemeda [18] discussed the consistent behavior of HPM for frictional order linear and non-linear partial differentia equations. Moreover, the related work with this article can be seen in [19-23].

Basic equation

The MHD flow of incompressible fluid is based on the Darcy's law, continuity, momentum and heat equations for third grade fluid given by:

$$r = -\left[\mu + \alpha_1 \frac{\partial}{\partial t}\right] \frac{\phi \boldsymbol{v}}{\mathcal{K}_1}, \tag{1}$$

$$\nabla \cdot \vec{v} = 0 \tag{2}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + div\boldsymbol{\tau} + \rho g sin\theta + \vec{\mathbf{J}} \times \vec{\mathbf{B}} + r \tag{3}$$

$$\rho c_p \frac{D\theta}{Dt} = k\nabla^2 \theta + tr\left(\mathbf{T} \cdot \mathbf{L}\right) \tag{4}$$

where D/Dt is the material time derivative, ρ – the fluid density, $\vec{v}$ is – the velocity vector of the fluid, g$\sin\theta$ – the external body force, $\vec{\mathbf{J}} \times \vec{\mathbf{B}} = \left[0, \sigma {B_0}^2 v(y), 0\right]$ – the Lorentz force per unit volume, $\vec{\mathbf{J}} = \sigma(E + v \times \vec{\mathbf{B}})$ is the current density, $\vec{\mathbf{B}} = (0, \mathrm{B}_0, 0)$ – the uniform magnetic filed, σ – the electrical conductivity, μ – the dynamic viscosity, θ – the temperature, k – the thermal conductivity, c_p – the specific heat, ϕ – the porousity, $\mathcal{K}_1$ – the Darcy permeability, and $\mathbf{T}$ – the Caushy stress tensor is define for third grade fluid as:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1 \tag{5}$$

$$\mathbf{A}_1 = L + L^T, \quad L = \text{grad}\boldsymbol{v} \tag{6}$$

$$\mathbf{A}_\text{n} = \frac{DA_{\text{n}-1}}{Dt} + \mathbf{A}_{\text{n}-1}L + L^T\mathbf{A}_{\text{n}-1}, \ \ n \geq 1 \tag{7}$$

where $p\mathbf{I}$ is the isotropic stress, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the Rivlin Ericksen stress tensor, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are the meterial constant. For third grade fluid:

$$\mu \geq 0, \ \alpha_1 \ \ \geq 0, \ \beta_1 = \beta_2 = 0, \ \beta_3 \geq 0, \qquad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}$$

Equations (1-8) are related with modified Darcy's law mentioned in [9, 10]

Formulation of problem

Consider a wide porous inclined belt. We consider a thin layer of third order fluid of uniform thickness δ draining down the belt. The belt is oscillating and the fluid drains down the belt due to gravity. Uniform magnetic field is applied to the belt transversely. The *x*-axis is taken parallel to the belt and *y*-axis is perpendicular to the belt. Assuming that the flow is laminar and unsteady, ambient air pressure is absent whereas the fluid shear forces keep gravity balanced and the thickness of the film remains constant.

The velocity and temperature field yields the form:

$$V = [v(y,t), 0, 0], \text{and } \ \theta = \theta(y,t) \tag{8}$$

The flow is under the following oscillating boundary conditions:

$$v(0,t) = V\cos\omega t, \qquad \frac{\partial \boldsymbol{v}(\delta,t)}{\partial y} = 0 \tag{9}$$

$$\theta(0,t) = \theta_0, \quad \theta(\delta,t) = \theta_1 \tag{10}$$

where ω is the frequency of the oscillating belt and θ denote temperature.

According to the previous assumptions the momentum and energy eqs. (3) and (4) yield the form:

$$\rho\frac{\partial v}{\partial t} = \frac{\partial}{\partial y}T_{yx} + \rho g\sin\theta - \sigma {B_0}^2 v + r \ , \tag{11}$$

$$\rho c_\text{p}\frac{\partial \theta}{\partial t} = k\left(\frac{\partial^2\theta}{\partial y^2}\right) + T_{yx}\left(\frac{\partial v}{\partial y}\right), \tag{12}$$

The Cauchy stress component, T_{xy}, of the third order fluid is:

$$T_{xy} = \mu\frac{\partial \boldsymbol{v}}{\partial y} + \alpha_1\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + \beta_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial y}\right) + 2(\beta_2 + \beta_3)\left(\frac{\partial v}{\partial y}\right)^3 = T_{yx}, \tag{13}$$

Inserting eq. (13) in to eqs. (11) and (12) the momentum and energy equations are reduced:

$$\rho\frac{\partial v}{\partial t} = \mu\frac{\partial^2 v}{\partial y^2} + \rho\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right) + 6\beta_3\left(\frac{\partial v}{\partial y}\right)^2\left(\frac{\partial^2 v}{\partial y^2}\right) - \ \rho g\sin\theta - \sigma B_0 v + r \ , \tag{14}$$

$$\rho c_p\left(\frac{\partial \theta}{\partial t}\right) = k\left(\frac{\partial^2\theta}{\partial y^2}\right) + \mu\left(\frac{\partial v}{\partial y}\right)^2 + \alpha_1\frac{\partial v}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + 2\beta_3\left(\frac{\partial v}{\partial y}\right)^4, \tag{15}$$

Introducing non-dimensional variables:

$$\tilde{v}=\frac{v}{V}\ ,\ \tilde{y}=\frac{y}{\delta},\ \ \tilde{t}=\frac{t\mu}{\delta^2\rho}\ ,\ \alpha=\frac{\alpha_1}{\rho\delta^2}\ ,\ \beta=\frac{\beta_3V^2}{\mu\delta^2}\ ,\ S_{\mathrm{t}}=\frac{\rho\delta^2 g\sin\theta}{\mu V}\ ,\ M=\frac{\delta^2\sigma\mathrm{B}_0}{\mu V}\ ,\ \Phi=\frac{\delta^2\xi}{\mathcal{K}_1},$$
$$B_r=\frac{V^2\mu}{k(\theta_1-\theta_0)},\ P_r=\frac{c_p\mu}{k},\ \bar{\theta}=\frac{\theta-\theta_0}{\theta_1-\theta_0},\gamma_1=\frac{1}{1+\Phi_1},\ \gamma_2=\frac{\alpha}{1+\alpha\Phi_1},\gamma_3=\frac{6\beta}{1+\alpha\Phi_1},\gamma_4=$$
$$=\frac{M+\Phi_1}{1+\alpha\Phi_1},\ \ \gamma_5=\frac{2\beta\Phi_1}{1+\alpha\Phi_1},\gamma_6=\frac{S_t}{1+\alpha\Phi_1},\qquad \mathrm{vv} \tag{16}$$

where M is the magnetic parameter, β – the non-Newtonian parameter, S_{t} – the stock number, and $\alpha,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6$ are the non-dimensional variables.

Inserting eq. (16) into eqs. (14) and (15), we obtain:

$$\frac{\partial v}{\partial t}=\gamma_1\frac{\partial^2 v}{\partial y^2}+\gamma_2\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right)+\gamma_3\left(\frac{\partial v}{\partial y}\right)^2\left(\frac{\partial^2 v}{\partial y^2}\right)-\gamma_4 v-\gamma_5 v\left(\frac{\partial v}{\partial y}\right)^2-\gamma_6 \tag{17}$$

$$\frac{\partial\theta}{\partial t}=\frac{1}{P_r}\frac{\partial^2\theta}{\partial y^2}+\frac{B_r}{P_r}\left[\left(\frac{\partial v}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial^2 v}{\partial t\partial y}\right)+2\beta_3\left(\frac{\partial v}{\partial y}\right)^4\right] \tag{18}$$

The appropriate oscillating boundary conditions are reduced:

$$v(o,t)=\ \cos\omega t,\qquad \frac{\partial v(1,t)}{\partial y}=0 \tag{19}$$

$$\theta(0,t)=0,\quad \theta(1,t)=1 \tag{20}$$

The OHAM solution

In this section, we apply OHAM method on eqs. (17) and (18) together with boundary condition (19) and (20) and study zero, first and second component problems.

Zero and first component problems of velocity and temperature profiles are:

$$p^0:\quad \frac{\partial^2 v_0(y,t)}{\partial y^2}=\frac{-\gamma_6}{\gamma_1} \tag{21}$$

$$\frac{\partial^2\theta_0(y,t)}{\partial y^2}=0 \tag{22}$$

$$p^1:\quad \frac{\partial^2 v_1(y,t)}{\partial y^2}\gamma_1=-c_1\frac{\partial v_0}{\partial t}-\gamma_4 c_1 v_0+\gamma_6+c_1\gamma_6+c_1 v_0\gamma_5\left(\frac{\partial v_0}{\partial \mathrm{y}}\right)^2+\gamma_1\frac{\partial^2 v_0}{\partial y^2}(1+c_1)+$$
$$+c_1\gamma_3\left(\frac{\partial v_0}{\partial y}\right)^2\left(\frac{\partial^2 v_0}{\partial y^2}\right)+c_1\gamma_2\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial^2 v_0}{\partial y^2}\right) \tag{23}$$

$$\frac{\partial^2\theta_1}{\partial y^2}=-P_r c_3\frac{\partial\theta_0}{\partial t}+B_r c_3\left[\left(\frac{\partial v_0}{\partial y}\right)^2+2\beta\left(\frac{\partial v_0}{\partial y}\right)^4+\alpha\frac{\partial v_0}{\partial y}\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial v_0}{\partial y}\right)\right]+\frac{\partial^2\theta_0}{\partial y^2}(1+c_3) \tag{24}$$

Solutions of eqs. (21- 24) using boundary conditions (19, 20) are:

$$v_0(y,t)=\cos[t\omega]-\left[\cos[t\omega]-\frac{\gamma_6}{2\gamma_1}\right]\mathrm{y}-\left[\frac{\gamma_6}{2\gamma_1}\right]\mathrm{y}^2 \tag{25}$$

$$\theta_0(y,t) = y \tag{26}$$

$$v_{11} = \frac{(4\gamma_3-\gamma_5)\cos[t\omega]^2\gamma_6}{8\gamma_1^2} - \frac{\omega\sin[t\omega]}{3\gamma_1} + \frac{\left(\gamma_4+\gamma_5\cos[t\omega]^2\right)\cos[t\omega]}{3\gamma_1} + \frac{\gamma_4\gamma_6}{24\gamma_1^2} - \frac{(20\gamma_3+3\gamma_5)\gamma_6^2\cos[t\omega]}{120\gamma_1^3} + \frac{(20\gamma_3+\gamma_5)\gamma_6^3}{480\gamma_1^4} \tag{27}$$

$$v_{12} = \frac{\omega\mathrm{Sin}[t\omega]}{2\gamma_1} - \frac{\left(\gamma_4+\gamma_5\cos[t\omega]^2\right)\cos[t\omega]}{2\gamma_1} - \frac{(\gamma_3+\gamma_5)\gamma_6\cos[t\omega]^2}{2\gamma_1^2} + \frac{(4\gamma_3-\gamma_5)\gamma_6^2\cos[t\omega]}{2\gamma_1^3} - \frac{\gamma_3\gamma_6^3}{8\gamma_1^4} \tag{28}$$

$$v_{13} = \frac{\left(2\gamma_1\gamma_4+2\gamma_5\gamma_1\cos[t\omega]^2-7\gamma_5\gamma_6\cos[t\omega]\right)\cos[t\omega]}{12\gamma^2{}_1} - \frac{\omega\sin[t\omega]}{6\gamma_1} - \frac{(8\gamma_3+7\gamma_5)\cos[t\omega]\gamma_6^2}{24\gamma_1^3} - \frac{\gamma_4\gamma_6}{12\gamma_1^2} + \frac{(8\gamma_3-\gamma_5)\gamma_6^3}{48\gamma_1^4} \tag{29}$$

$$v_{14} = \frac{\gamma_4\gamma_6}{24\gamma_1^2} + \frac{\cos[t\omega]^2\gamma_5\gamma_6}{24\gamma_1^2}(5\cos[t\omega]-7) - \frac{\gamma_6^3}{12\gamma_1^4}\left(\gamma_3+\frac{5\gamma_5}{8}\right) \tag{30}$$

$$v_{15} = \frac{\cos[t\omega]\gamma_5\gamma_6^2}{10\gamma_1^3} - \frac{\gamma_5\gamma_6^3}{20\gamma_1^4}, \tag{31}$$

$$v_{16} = \frac{\gamma_5\gamma_6^3}{60\gamma_1^4} \tag{32}$$

$$v_1(y,t) = c_1[v_{11}y + v_{12}y^2 + v_{13}\mathrm{y}^3 + v_{14}y^4 + v_{15}y^5 + v_{16}y^6] \tag{33}$$

$$\theta_{11} = \frac{1}{4}\cos[t\omega](2\alpha\omega\sin[t\omega]-\beta\cos[3t\omega]) - \frac{(2+3\beta)\cos[t\omega]^2}{4} + \frac{(1+3\beta)\gamma_6\cos[t\omega]}{6\gamma_1} + \frac{\beta\gamma_6\cos[3t\omega]}{6\gamma_1} - \\ - \frac{\alpha\omega\sin[t\omega]\gamma_6}{12\gamma_1} - \frac{(1+6\beta)\gamma_6^2}{24\gamma_1^2} - \frac{\beta\cos[2t\omega]\gamma_6^2}{4\gamma_1^2} + \frac{\beta\cos[t\omega]\gamma_6^3}{10\gamma_1^3} - \frac{c_3\gamma_6^4}{80\gamma_1^4} \tag{34}$$

$$\theta_{12} = \frac{(2+3\beta)\cos[t\omega]^2}{2} + \frac{1}{4}\cos[t\omega](\beta\cos[3t\omega]-2\alpha\omega\mathrm{con}[t\omega]) - \frac{(1-3\beta)\gamma_6\cos[t\omega]}{2\gamma_1} - \frac{\beta\cos[3t\omega]\gamma_6}{2\gamma_1} + \frac{\alpha\omega\sin[t\omega]\gamma_6}{4\gamma_1} + \\ + \frac{\beta\gamma_6^4}{16\gamma_1^4} + \frac{(1+6\beta)\gamma_6^2}{8\gamma_1^2} + \frac{3\beta\cos[2t\omega]\gamma_6^2}{4\gamma_1^2} - \frac{\beta\cos[t\omega]\gamma_6^3}{2\gamma_1^3} \tag{35}$$

$$\theta_{13} = \frac{(1+3\beta)\cos[t\omega]\gamma_6}{3\gamma_1} + \frac{\beta\cos[3t\omega]\gamma_6}{3\gamma_1} - \frac{\alpha\omega\sin[t\omega]\gamma_6}{6\gamma_1} - \frac{(1+6\beta)\gamma_6^2}{6\gamma_1^2} - \frac{\beta\cos[2t\omega]\gamma_6^2}{\gamma_1^2} + \frac{\beta\cos[t\omega]\gamma_6^3}{\gamma_1^3} - \frac{\beta\gamma_6^4}{6\gamma_1^4} \tag{36}$$

$$\theta_{14} = \frac{(1+6\beta)\gamma_6^2}{12\gamma_1^2} + \frac{\beta\cos[2t\omega]\gamma_6^2}{2\gamma_1^2} - \frac{\beta\cos[t\omega]\gamma_6^3}{\gamma_1^3} + \frac{\beta\gamma_6^4}{4\gamma_1^4} \tag{37}$$

$$\theta_{15} = \frac{2\beta\cos[t\omega]\gamma_6^3}{5\gamma_1^3} - \frac{\beta\gamma_6^4}{5\gamma_1^4} \tag{38}$$

$$\theta_{16} = \frac{\beta\gamma_6^4}{15\gamma_1^4} \tag{39}$$

$$\theta_1(y,t) = B_r c_3[\theta_{11}\mathrm{y} + \theta_{12}\mathrm{y}^2 + \theta_{13}\mathrm{y}^3 + \theta_{14}\mathrm{y}^4 + \theta_{15}\mathrm{y}^5 + \theta_{16}\mathrm{y}^6] \tag{40}$$

The solutions of second component of velocity and temperature fields are too large. Therefore, the numerical solutions are given up to first order while, graphical solutions are given up to second order.

The value of c_i for the velocity components are c_1 = –1.32285737, and c_2 = –1.15543843.

Also the values of c_i for the temperature distribution are:

$$c_1 = 0.02431759,\ c_2 = -0.10984471,\ c_3 = -1.88757428,\qquad c_4 = -0.59500373$$

The HPM solution

In this section we apply HPM method on eqs. (17) and (18) together with boundary conditions (19, 20) and study zero, first and second component problems.

Zero and first order problems are:

$$p^0:\quad \frac{\partial^2 v_0(y,t)}{\partial y^2}\gamma_1 + \gamma_6 = 0 \tag{41}$$

$$\frac{\partial^2 \theta_0(y,t)}{\partial y^2} = 0 \tag{42}$$

$$p^1:\quad \frac{\partial^2 v_1(y,t)}{\partial y^2}\gamma_1 = -\frac{\partial v_0}{\partial \mathrm{t}} - v_0\gamma_4 + 2\gamma_6 - v_0\gamma_5\left(\frac{\partial v_0}{\partial y}\right)^2 + 2\gamma_1\left(\frac{\partial^2 v_0}{\partial y^2}\right) + \gamma_3\left(\frac{\partial v_0}{\partial y}\right)^2\left(\frac{\partial^2 v_0}{\partial y^2}\right) + \gamma_2\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial^2 v_0}{\partial y^2}\right) \tag{43}$$

$$\frac{\partial^2 \theta_1(y,t)}{\partial y^2} = -P_r\frac{\partial \Theta_0}{\partial t} + B_r\left[\left(\frac{\partial v_0}{\partial y}\right)^2 + 2\beta\left(\frac{\partial v_0}{\partial y}\right)^4 + \alpha\frac{\partial v_0}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial v_0}{\partial y}\right)\right] + 2\frac{\partial^2 \theta_0}{\partial y^2} \tag{44}$$

Using boundary conditions (19) and (20) into eqs. (41)-(44), the component solutions are:

$$v_0(y,t) = \cos[t\omega] - \left[\cos[t\omega] - \frac{\gamma_6}{2\gamma_1}\right]\mathrm{y} - \left[\frac{\gamma_6}{2\gamma_1}\right]\mathrm{y}^2 \tag{45}$$

$$\theta_0(y,t) = y \tag{46}$$

$$v_{11} = \frac{(4\gamma_3-\gamma_5)\gamma_6\cos[t\omega]^2}{8\gamma_1^2} - \frac{\omega\sin[t\omega]}{3\gamma_1} + \frac{(\gamma_4+\gamma_5\cos[t\omega]^2)\cos[t\omega]}{3\gamma_1} + \frac{\gamma_4\gamma_6}{24\gamma_1^2} - \frac{(20\gamma_3+3\gamma_5)\gamma_6^2\cos[t\omega]}{120\gamma_1^3} + \frac{(20\gamma_3+\gamma_5)\gamma_6^3}{480\gamma_1^4} \tag{47}$$

$$v_{12} = \frac{\omega\sin[t\omega]}{2\gamma_1} - \frac{(\gamma_4+\cos[t\omega]^2\gamma_5)\cos[t\omega]}{2\gamma_1} - \frac{(\gamma_3+\gamma_5)\cos[t\omega]^2\gamma_6}{2\gamma_1^2} + \frac{(4\gamma_3-\gamma_5)\gamma_6^2\cos[t\omega]}{8\gamma_1^3} - \frac{\gamma_3\gamma_6^3}{8\gamma_1^4} \tag{48}$$

$$v_{13} = \frac{\cos[t\omega]}{6\gamma_1} - \frac{\omega(\gamma_4+\cos[t\omega]^2\gamma_5)\mathrm{Sin}[t\omega]}{6\gamma_1} - \frac{7\cos[t\omega]^2\gamma_5\gamma_6}{12\gamma_1^2} - \frac{(8\gamma_3+7\gamma_5)\gamma_6^2\cos[t\omega]}{24\gamma_1^3} - \frac{\gamma_4\gamma_6}{12\gamma_1^2} + \frac{(8\gamma_3-\gamma_5)\gamma_6^3}{48\gamma_1^4} \tag{49}$$

$$v_{14} = \frac{5\cos[t\omega]^2\gamma_5\gamma_6}{24\gamma_1^2} - \frac{7\cos[t\omega]\gamma_5\gamma_6^2}{24\gamma_1^3} + \frac{\gamma_4\gamma_6}{24\gamma_1^2} - \frac{\gamma_6^3}{12\gamma_1^4}(8\gamma_3 + 5\gamma_5) \tag{50}$$

$$v_{15} = \frac{\cos[t\omega]\gamma_5\gamma_6^2}{10\gamma_1^3} - \frac{\gamma_5\gamma_6^3}{20\gamma_1^4} \tag{51}$$

$$v_{16} = \frac{\gamma_5\gamma_6^3}{60\gamma_1^4} \tag{52}$$

$$v_1(y,t) = v_{11}\mathrm{y} + v_{12}\mathrm{y}^2 + v_{13}\mathrm{y}^3 + v_{14}\mathrm{y}^4 + v_{15}\mathrm{y}^5 + v_{16}\mathrm{y}^6 \tag{53}$$

$$\theta_{11} = \frac{\cos[t\omega](2\alpha\omega\sin[t\omega]-\beta\cos[3t\omega])}{4} - \frac{\cos[t\omega]^2(2-3\beta)}{4} + \frac{(5\gamma_1^2+15\gamma_1^2\beta+3\beta)\gamma_6\cos[t\omega]}{6\gamma_1^3} + \frac{\beta\cos[3t\omega]\gamma_6}{6\gamma_1} - \\ - \frac{\alpha\omega\sin[t\omega]\gamma_6}{12\gamma_1} - \frac{\beta\cos[2t\omega]\gamma_6^2}{4\gamma_1^2} + \frac{(60\gamma_1^2\beta+3\beta\gamma_6^2)\gamma_6^2}{240\gamma_1^4} \tag{54}$$

$$\theta_{12} = \frac{\cos[t\omega]^2(2+3\beta)}{4} + \frac{\beta\cos[t\omega](\cos[3t\omega]-2\alpha\omega\sin[t\omega])}{4} - \frac{(1-3\beta)\gamma_6\cos[t\omega]}{2\gamma_1} - \frac{\beta\cos[3t\omega]\gamma_6}{2\gamma_1} + \frac{\alpha\omega\sin[t\omega]\gamma_6}{4\gamma_1} + \\ + \frac{3\beta\cos[2t\omega]\gamma_6^2}{4\gamma_1^2} - \frac{\beta\cos[t\omega]\gamma_6^3}{2\gamma_1^3} + \frac{(2\gamma_1^2+12\gamma_1^2\beta+\beta\gamma_6^2)\gamma_6^2}{8\gamma_1^4} \tag{55}$$

$$\theta_{13} = \frac{(1+3\beta)\gamma_6\cos[t\omega]}{3\gamma_1} + \frac{\beta\gamma_6\cos[3t\omega]}{3\gamma_1} - \frac{\alpha\omega\gamma_6\sin[t\omega]}{6\gamma_1} + \frac{\beta\gamma_6^3\cos[t\omega]}{\gamma_1^3} - \frac{\beta\gamma_6^2\cos[2t\omega]}{\gamma_1^2} - \frac{(\gamma_1^2+6\gamma_1^2\beta+\beta\gamma_6^2)\gamma_6^2}{6\gamma_1^4} \tag{56}$$

$$\theta_{14} = \frac{\beta\gamma_6^2\mathrm{Cos}[2t\omega]}{2\gamma_1^2} - \frac{\beta\gamma_6^3\mathrm{Cos}[t\omega]}{\gamma_1^3} + \frac{(\gamma_1^2+6\gamma_1^2\beta+3\beta\gamma_6^2)\gamma_6^2}{12\gamma_1^4} \tag{57}$$

$$\theta_{15} = \frac{2\beta\cos[t\omega]\gamma_6^3}{5\gamma_1^3} - \frac{\beta\gamma_6^4}{5\gamma_1^4} \tag{58}$$

$$\theta_{16} = \frac{\beta\gamma_6^4}{15\gamma_1^4} \tag{59}$$

$$\theta_1(y,t) = B_r[\theta_{11}\mathrm{y} + \theta_{12}\mathrm{y}^2 + \theta_{13}\mathrm{y}^3 + \theta_{14}\mathrm{y}^4 + \theta_{15}\mathrm{y}^5 + \theta_{16}\mathrm{y}^6] \tag{60}$$

The solution of second component of velocity and temperature distribution is too large. Therefore, the expressions of solutions are given up to first order while graphical solutions are given up to second order

Results and discussion

In this paper, we examined the approximate analytical solutions for both velocity and temperature distribution of unsteady MHD thin film flow of non-Newtonian fluid through porous and oscillating inclined belt. The arising non-linear partial differential equations are solved using OHAM and HPM methods. The results of both methods are compared numerically and graphically for velocity and temperature distribution. The results obtained from OHAM and HPM are in excellent agreement. In tabs. 1 and 2 we investigated the numerical comparisons of these methods along with absolute error at different time level. Figure 1 shows

Table 1. Comparison of OHAM and HPM for the velocity profile, by taking, ω = 0.2, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6, t = 5.5

x	OHAM	HPM	Absolute error
0.0	0.453.596121	0.453596121	0
0.1	0.425637118	0.421495307	$4.1418.8 \cdot 10^{-3}$
0.2	0.392523074	0.388479329	$4.04337 \cdot 10^{-3}$
0.3	0.354947252	0.353735103	$1.2121 \cdot 10^{-3}$
0.4	0.313506318	0.316455255	$2.9489 \cdot 10^{-3}$
0.5	0.268679531	0.2758536470	$7.1741 \cdot 10^{-3}$
0.6	0.220809170	0.2311857844	$1.0376 \cdot 10^{-2}$
0.7	0.170082725	0.1817744049	$1.1691 \cdot 10^{-2}$
0.8	0.116517087	0.127040376	$1.0523 \cdot 10^{-2}$
0.9	0.059944556	0.06653880	$6.5942 \cdot 10^{-3}$
1.0	$7.771 \cdot 10^{-16}$	$-9.378 \cdot 10^{-17}$	$8.7430 \cdot 10^{-16}$

Table 2. Comparison of OHAM and HPM for the heat distribution, by taking, ω = 0.2, α = 0.02, Pr = 0.6, Br = 4, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6, β = 0.5, t = 7

x	OHAM	HPM	Absolute error
0.0	0	0	0
0.1	0.105885889	0.095952644	$9.9332 \cdot 10^{-3}$
0.2	0.211560411	0.191806597	$1.9753 \cdot 10^{-2}$
0.3	0.316856482	0.287730310	$2.9126 \cdot 10^{-2}$
0.4	0.421457769	0.383956928	$3.7500 \cdot 10^{-2}$
0.5	0.524909829	0.480826884	$4.4082 \cdot 10^{-2}$
0.6	0.626627403	0.578830688	$4.7796 \cdot 10^{-2}$
0.7	0.725898232	0.678654998	$4.7243 \cdot 10^{-2}$
0.8	0.821883792	0.781234424	$4.0649 \cdot 10^{-2}$
0.9	0.913617376	0.887810727	$2.5806 \cdot 10^{-2}$
1.0	0.999999999	1.000000042	$5.5615 \cdot 10^{-15}$

the geometry of the problem. Figures 2 and 3 shows the graphical comparison of OHAM and HPM solutions at different values of physical parameters. Figures 4 and 5 give the influence of different time level on velocity and temperature distribution. Figures 6 and 7 show the velocity and temperature distribution by taking the different values of the physical domain oscillates with the belt oscillation. Increasing the non-Newtonian parameter, β, of the third order fluid causes more thickening of the boundary layer. In general, for the case with suction through the porous belt the Newtonian thin layer is thinner than that for the third order fluid layer. For a blowing through the porous belt, the fluids behave opposite to the case of suction. In case of Increasing, if the blowing velocity increases for the Newtonian fluid, then the shear boundary layer of the Newtonian fluid layer becomes thicker quickly, whilst for the third order fluids the thickness of boundary layers is not so thinner to variations in the blowing and only small thinning of the boundary layer happens. The effects of various non-dimensional physical parameters are discussed in figs. 8 to 9 for velocity and temperature distribution. Figures 8 and 9 show the variation in γ_1 on velocity and temperature distribution. It is revealed that velocity and temperature fields decrease by increasing γ_1. Moreover, the influence of the model parameters γ_2 and γ_3 are given in figs. (10)-(13), respectively. These figures illustrate that γ_2 and γ_3 have opposite roles on the velocity and temperature fields. These figures show that velocity field increases for large values of γ_2 whereas temperature field is unchanged and velocity decreases for increasing γ_3.

It is clear from these figures that for $\gamma_2 < 1$ and $\gamma_3 < 1$, the fluids tend to Newtonian status. Further, it is clear from figs. (14)-(19) that due to the friction of no-slip boundary fluid along with the belt renders oscillation in the same stage and the amplitude. The amplitude of the velocity decreases rapidly towards the free surface of the inclined belt.

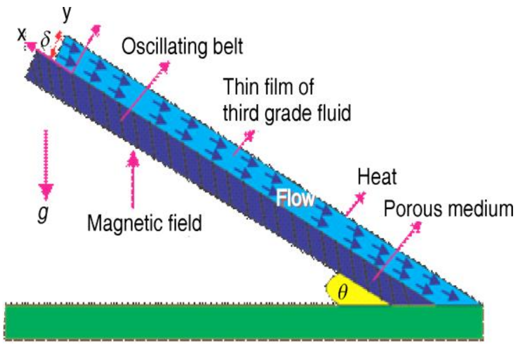
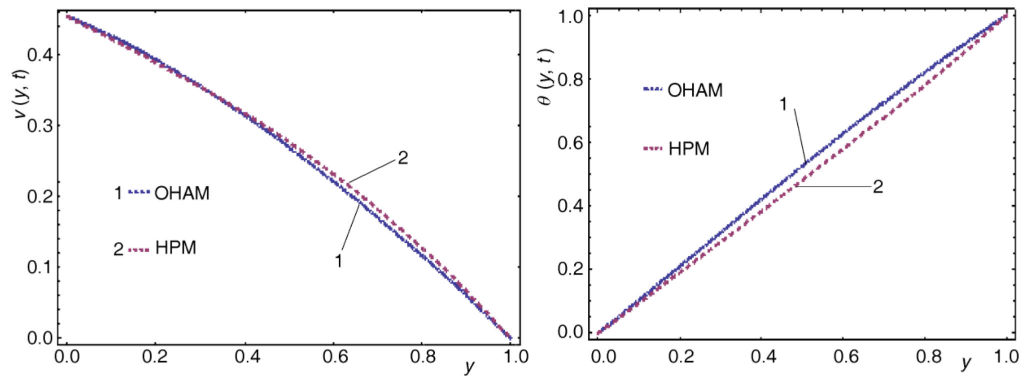
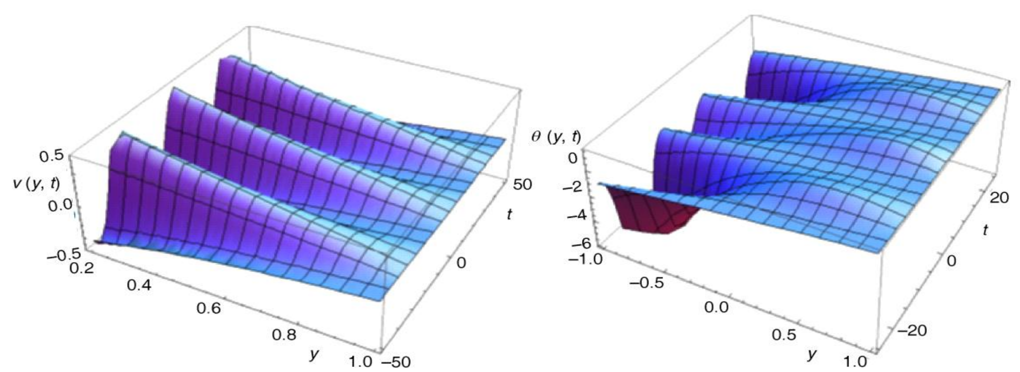


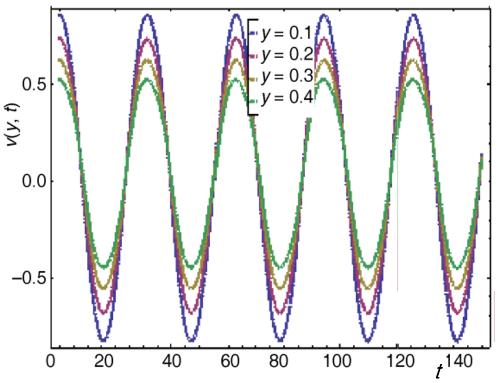
Figure 1. Geometry of the problem

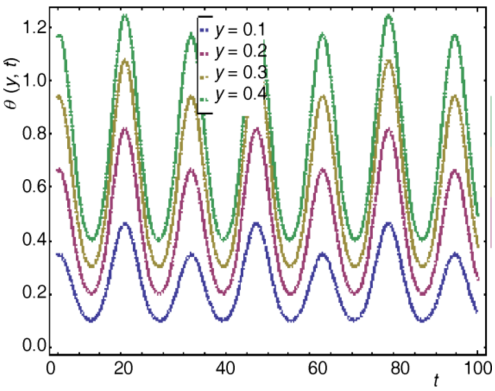


Figures 2 and 3. Comparison of OHAM and HPM methods for velocity profile (on the left) by taking ω = 0.2, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6, t = 5.5 and temperature distribution (on the right) by taking ω = 0.2, α = 0.02, Pr = 0.6, Br = 4, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6, β = 0.5, t = 7

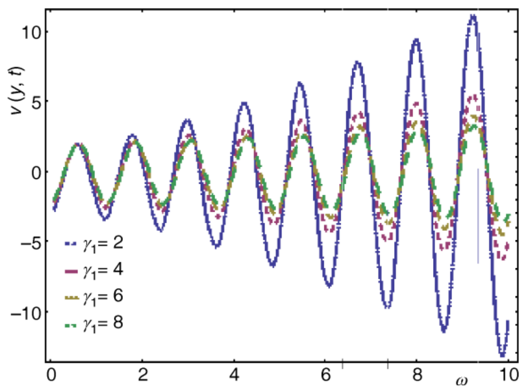


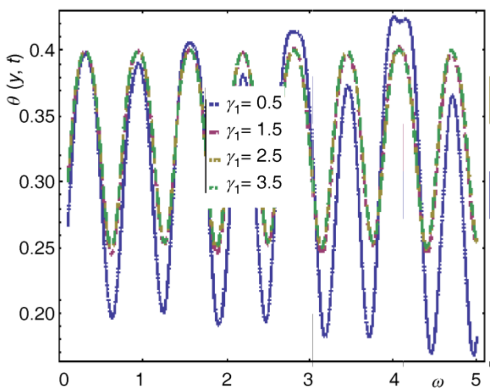
Figures 4 and 5. Influence of different time level on velocity profile (on the left) and temperature distribution (on the right)



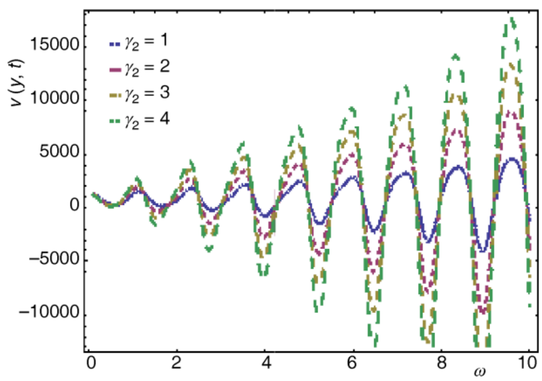


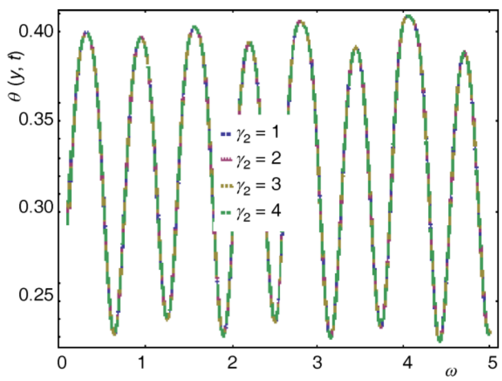
Figures 6 and 7. Velocity distribution *t* of fluid at various time level (on the left) when ω = 0.2, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6 and temperature distribution of fluid (on the right) by taking ω = 0.2, α = 0.02, Pr = 0.6, Br = 4, γ_3 = 0.1, γ_4 = 0.2, γ_6 = 0.3, γ_5 = 0.4, γ_2 = 0.5, γ_1 = 0.6, β = 0.5



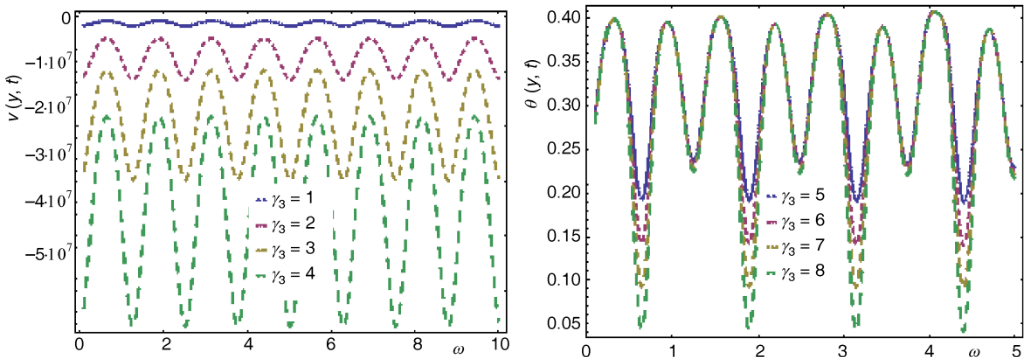


Figures 8 and 9. Effect of γ_1 on the velocity profile (on the left) when γ_5= 0.2, γ_4 = 0.2, γ_3 = 0.1, γ_2 = 0.2, γ_6 = 0.3, γ = 3, *t* = 5, Φ = 0.2 and on temperature distribution (on the right) by taking. γ_5 = 0.2, γ_4 = 0.5, γ_3 = 0.4, γ_2 = 0.8, γ_6 = 0.5, γ = 0.4, *t* = 5, β = 0.6, Br = 0.5, α = 0.03

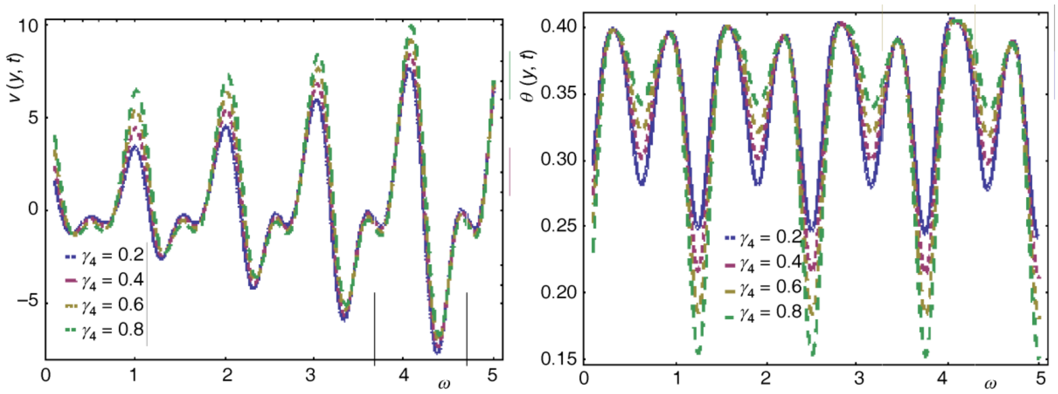




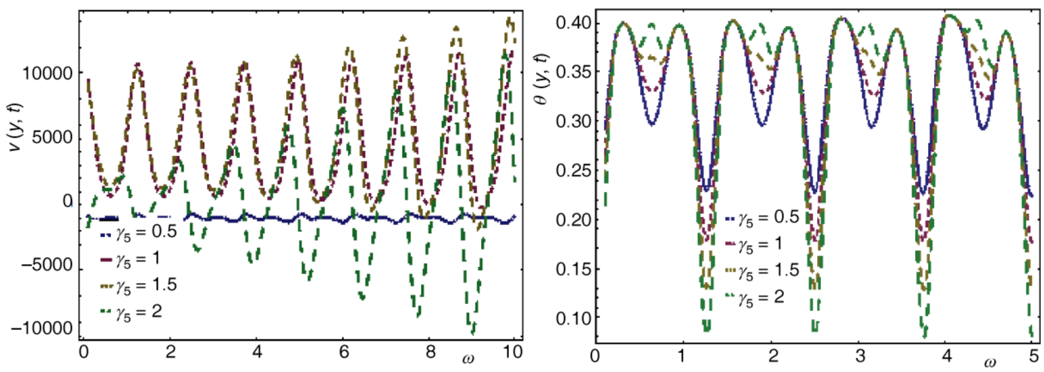
Figures 10 and 11. Effect of γ_2 on the velocity (on the left) when γ_5= 0.3, γ_4 = 0.4, γ_3 = 0.2, γ_1 = 0.2, γ_6 = 0.5, γ = 3, *t* = 5, Φ = 0.4 and temperature distribution (on the right) by taking γ_5 = 0.2, γ_4 = 0.5, γ_3 = 0.4, γ_1 = 0.8, γ_6 = 0.5, γ = 0.4, *t* = 5, β = 0.6, Br = 0.5, α = 0.03



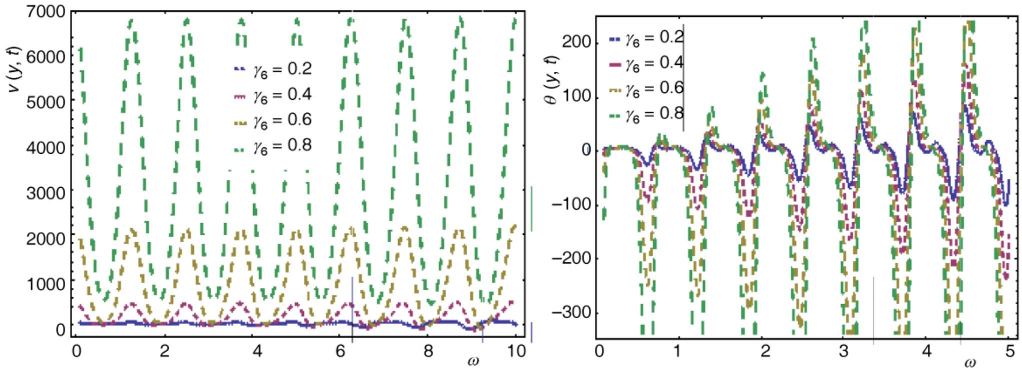
Figures 12 and 13. Effect of γ_3 on the velocity (on the left) when γ_4= 0.4, γ_1 = 0.1, γ_2 = 0.6, γ_6 = 0.4, γ = 3, t = 5, γ_5 = 0.3, Φ = 0.4 and temperature distribution (on the right) by taking γ_5 = 0.2, γ_4 = 0.5, γ_2 = 0.4, γ_1 = 0.8, γ_6 = 0.5, γ = 0.4, t = 5, β = 0.6, Br = 0.5, α = 0.03



Figures 14 and 15. Effect of γ_4 on the velocity (on the left) when γ_3= 0.1, γ_1 = 0.2, γ_2 = 0.6, γ_6 = 0.4, γ = 3, t = 5, γ_5 = 0.2, Φ = 0.5 and temperature distribution (on the right) by taking γ_5 = 0.3, γ_3 = 0.5, γ_2 = 0.3, γ_1 = 0.7, γ_6 = 0.4, γ = 0.4, t = 5, β = 0.6, Br = 0.5, α = 0.04



Figures 16 and 17. Effect of γ_5 on the velocity (on the left) when γ_4= 0.4, γ_1 = 0.1, γ_2 = 0.6, γ_6 = 0.4, γ = 3, t = 5, γ_5 = 0.3, Φ = 0.4 and temperature distribution (on the right) by taking γ_5 = 0.2, γ_4 = 0.5, γ_2 = 0.4, γ_1 = 0.8, γ_6 = 0.5, γ = 0.4, t = 5, β = 0.6, Br = 0.5, α = 0.03



Figures 18 and 19. Effect of γ_6 on the velocity (on the left) when $\gamma_4 = 0.5$, $\gamma_1 = 0.6$, $\gamma_2 = 0.6$, $\gamma_3 = 0.4$, $\gamma = 3$, $t = 5$, $\gamma_5 = 0.3$, $\Phi = 0.6$ and temperature distribution (on the right) by taking $\gamma_5 = 0.2$, $\gamma_4 = 0.5$, $\gamma_2 = 0.4$, $\gamma_1 = 0.8$, $\gamma_3 = 0.5$, $\gamma = 0.4$, $t = 5$, $\beta = 0.6$, Br = 0.5, $\alpha = 0.03$

Conclusion

Heat transferred analysis and unsteady thin film flow of an MHD third order fluid through a porous and oscillating belt has been discussed. The problems have been solved OHAM and HPM. Solutions at different time levels on velocity and temperature fields have been shown by taking the different values of the physical parameters. The results obtained from both methods are very identical. The effect of different nondimensional physical parameters are plotted and discussed.

Nomenclature

A$_1$-**A**$_3$ – kinematical tensors, [Nm^{-2}]
Br – Brinkman number, [–]
c_p – specific heat, [kg^{-1}K^{-1}]
g – gravitational acceleration, [ms^{-2}]
$\mathcal{K}_1$ – Darcy permeability, [–]
M – magnetic parameter, [am^{-1}]
Pr – Prandtl number, [–]
p**I** – isotropic stress, [Nm^{-2}]
$-p$**I** – spherical stress, [Nm^{-2}]
St – Stock number, [–]
T – Cauchy stress tensor, [Nm^{-2}]
$\vec{v}$ – velocity vector, [ms^{-1}]

Greek symbols

$\alpha_i(i = 1,2)$, $\beta_j(j = 1,2,3)$ – material constants
β – non-Newtonian parameter
γ_1-γ_6 – non-dimensional parameters
δ – constant thickness, [m]
θ – temperature distribution, [K]
κ – thermal conductivity
μ – dynamic viscosity, [kgm^{-1}s^{-1}]
ρ – constant density, [kgm^{-3}]
Φ – non-dimensional porousity parameter
ϕ – porousity, [–]
ωt – frequency of the oscillating belt, [s^{-1}]

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