A NEW COUPLING SCHEDULE FOR SERIES EXPANSION METHOD AND SUMUDU TRANSFORM WITH AN APPLICATIONS TO DIFFUSION EQUATION IN FRACTAL HEAT TRANSFER

by

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In this paper, we first propose the new coupling schedule for series expansion method and Sumudu transform, which is called the local fractional Sumudu series expansion method. Adopting the proposed technology, we consider the diffusion equation in fractal heat transfer. The obtained result shows that the presented technology is easy, simple, efficient and accurate.

Key words: *diffusion equation, Sumudu series expansion method, fractal heat transfer, local fractional derivative*

Introduction

In this paper, we consider the diffusion equation in fractal heat transfer within local fractional time and space derivatives [1, 2]:

$$\Omega_t^{(\beta)}(x,\tau)-\Omega_x^{(2\beta)}(x,\tau)=0,\quad 0<\beta\le 1 \tag{1}$$

where

$$\Omega_\tau^{(\beta)}(x,\tau_0)=\frac{\partial^\beta\Omega(x,\tau)}{\partial\tau^\beta}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Delta^\beta[\Omega(x,\tau)-\Omega(x,\tau_0)]}{(\tau-\tau_0)^\beta} \tag{2}$$

with $\Delta^\beta[\Omega(x,\tau)-\Omega(x,\tau_0)]\cong\Gamma(1+\beta)\Delta[\Omega(x,\tau)-\Omega(x,\tau_0)]$. Many methods are utilized to deal with local fractional differentiable equations, such as VIM [3-6], DM [6, 7], LFLT [8, 9], LFFT [10, 11], LFLVIM [12, 13], LFSEM [14, 15] and so on. The local fractional Sumudu transform (LFST) of $f(x)$ of order β $(0<\beta\le 1)$ is defined as [16]:

$$LFS_\beta\{\omega(\tau)\}=\frac{1}{\Gamma(1+\beta)}\int_0^\infty E_\beta(-h^{-\beta}\tau^\beta)\frac{\omega(\tau)}{h^\beta}(\mathrm{d}\tau)^\beta,\quad 0<\beta\le 1 \tag{3}$$

where

$${}_{\tau_0}I_\tau^{(\beta)}\omega(\tau)=\frac{1}{\Gamma(1+\beta)}\int_{\tau_0}^{\tau}\omega(\tau)(\mathrm{d}\tau)^\beta=\frac{1}{\Gamma(1+\beta)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{j=N-1}\omega(\tau_j)(\Delta\tau)^\beta \tag{4}$$

with the partitions of the interval $[\tau,\tau_0]$ are (τ_j,τ_{j+1}), $j=0,...,N-1,\Delta\tau=\tau_{j+1}-\tau_j$.

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Main target of the article is to derive the local fractional Sumudu series expansion method (LFSSEXM) in order to deal with the diffusion eq. (1) in fractal heat transfer within local fractional time and space derivatives.

Analysis of the LFSSEM

We rewrite eq. (1) in local fractional differential operator form:

$$\Phi_\tau^{(\beta)} = \mathrm{H}_\beta \Phi \tag{5}$$

where H_β is a linear local operator with respect to x and $\Phi_\tau^{(\beta)} = \partial^\beta \Phi(x,\tau)/\mathrm{d}\tau^\beta$.

Considering a multi-term separated functions of independent variables *t* and *x*:

$$\Phi(x,\tau) = \sum_{i=0}^{\infty} \eta_i(\tau)\varphi_i(x) \tag{6}$$

where $\eta_i(\tau) = \tau^{i\beta}/\Gamma(1+i\beta)$ and $\varphi_i(x)$ are two local fractional continuous functions, we have:

$$\Phi(x,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \varphi_i(x) \tag{7}$$

Taking the LFLT of right and left of eqs. (5) and (7):

$$\Phi(h,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \varphi_i(h) \tag{8}$$

$$LFS_\beta[\Phi_\tau^{(\beta)}(x,\tau)] = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} t^{i\beta} \varphi_{i+1}(h) \tag{9}$$

$$LFS_\beta[\mathrm{H}_\beta \Phi(x,\tau)] = \mathrm{H}_\beta \left[\sum_{i=0}^{\infty} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \varphi_i(h) \right] = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} t^{i\beta} [\mathrm{H}_\beta \varphi_i(h)] \tag{10}$$

we have:

$$\sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} t^{i\beta} \varphi_{i+1}(h) = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} t^{i\beta} [\mathrm{H}_\beta \varphi_i(h)] \tag{11}$$

which yields the follow relationship:

$$\varphi_{i+1}(h) = (\mathrm{H}_\beta \varphi_i)(h) \tag{12}$$

In view of eq. (12), we get:

$$\Phi(h,\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \varphi_i(h) \tag{13}$$

where the convergent condition is given as:

$$\lim_{i\to\infty} \left[\frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \varphi_i(h) \right] = 0 \tag{14}$$

Hence, taking inverse LFST (see Appendix), the solution of eq. (1) reads:

$$\Phi(x,\tau) = LFS_{\beta}^{-1}[\Phi(h,\tau)] \tag{15}$$

An illustrative example

Let us consider an initial value in the form:

$$\Phi(x,0) = \sin_{\beta}(x^{\beta}) \tag{16}$$

Making use of eq. (12), we get:

$$\begin{cases} \varphi_{i+1}(h) = (\mathrm{H}_{\beta}\varphi_i)(h) \\ \varphi_0(h) = LFS_{\beta}\{\sin_{\beta}(x^{\beta})\} = \dfrac{h^{\beta}}{1+h^{2\beta}} \end{cases} \tag{17}$$

which leads to:

$$\varphi_0(h) = \frac{h^{\beta}}{1+h^{2\beta}} \tag{18}$$

$$\varphi_1(h) = \frac{1}{1+h^{2\beta}} \tag{19}$$

$$\varphi_2(h) = -\frac{h^{\beta}}{1+h^{2\beta}} \tag{20}$$

$$\varphi_3(h) = -\frac{1}{1+h^{2\beta}} \tag{21}$$

$$\varphi_4(h) = \frac{h^{\beta}}{1+h^{2\beta}} \tag{22}$$

and so on.

Therefore, we have:

$$\Phi_1(h,\tau) = \sum_{i=\text{even}}^{\infty} (-1)^{\text{even}} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \frac{h^{\beta}}{1+h^{2\beta}} \tag{23}$$

$$\Phi_2(h,\tau) = \sum_{i=\text{odd}}^{\infty} (-1)^{i+1} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \frac{1}{1+h^{2\beta}} \tag{24}$$

so that:

$$\Phi(h,\tau) = \sum_{i=\text{odd}}^{\infty} (-1)^{i+1} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \frac{1}{1+h^{2\beta}} + \sum_{i=\text{even}}^{\infty} (-1)^{\text{even}} \frac{\tau^{i\beta}}{\Gamma(1+i\beta)} \frac{h^{\beta}}{1+h^{2\beta}} \tag{25}$$

Hence, with help of the inverse LFST, we obtain the non-differentiable solution of diffusion eq. (1) arising in fractal heat transfer:

$$\begin{aligned}\Phi(x,\tau) &= LFS_\beta^{-1}[\Phi(h,\tau)] \\ &= LFS_\beta^{-1}\left[\sum_{i=\text{odd}}^{\infty}(-1)^{i+1}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\frac{1}{1+h^{2\beta}}+\sum_{i=\text{even}}^{\infty}(-1)^{\text{even}}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\frac{h^\beta}{1+h^{2\beta}}\right] \\ &= \sum_{i=\text{odd}}^{\infty}(-1)^{i+1}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\cos_\beta(a\tau^\beta)+\sum_{i=\text{even}}^{\infty}(-1)^{\text{even}}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\sin_\beta(a\tau^\beta)\end{aligned} \tag{26}$$

Conclusions

In this work, we first had pointed out the local fractional Sumudu series expansion method (LFSSEM), which is a coupling scheme of LFSEM with LFST. An illustrative example for the non-differentiable solution of diffusion equation arising in fractal heat transfer is shown. The obtained result illustrates that the presented method is easy, simple, efficient, and accurate to find the solutions to local fractional partial differential equations.

Appendix

The inverse LFST is given as [16]:

$$LFS_\beta^{-1}\{LFS_\beta[f(\tau)]\} = f(\tau), \quad 0<\beta\le 1$$

The properties of LFST are [16]:

$$LFS_\beta\{f(\tau)+g(\tau)\} = F_\beta(h)+G_\beta(h)$$

$$LFS_\beta\frac{d^{2\beta}f(\tau)}{d\tau^{2\beta}} = \frac{1}{h^{2\beta}}[LFS_\beta\{f(\tau)\}-f(0)-h^\beta f^{(\beta)}(0)]$$

$$LFS_\beta\{\sin_\beta(a\tau^\beta)\} = \frac{ah^\beta}{1+a^2h^{2\beta}}$$

$$LFS_\beta\{\cos_\beta(a\tau^\beta)\} = \frac{1}{1+a^2h^{2\beta}}$$

Nomenclature

x – space co-ordinates, [m]
LFS_β – LFST operator, [–]
LFS_β^{-1} – inverse LFST operator, [–]

Greek symbols

β – time fractal dimensional order, [–]
τ – time, [s]
$\Phi(x, \tau)$ – concentration, [–]

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