AN APPROXIMATE METHOD FOR SOLVING A MELTING PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

by

Liang-Hui QU [a*]***, Lin XING*** [a]***, Zhi-Yun YU*** [a]***, Feng LING*** [b]***, and Jian-Guo XU*** [a]

[a] College of Science, Zhongyuan University of Technology, Zhengzhou, China
[b] School of Mathematics and Statistics, Zhaoqing University, Zhaoqing, China

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An effective thermal diffusivity method is used to solve one-dimensional melting problem with periodic boundary conditions in a semi-infinite domain. An approximate analytic solution showing the functional relation between the location of the moving boundary and time is obtained by using Laplace transform. The evolution of the moving boundary and the temperature field in the phase change domain are simulated numerically, and the numerical results are compared with previous results in open literature.

Key words: *melting problem, periodic boundary, Laplace transform, approximate analytic solution, moving boundary*

Introduction

The Stefan problem or the moving boundary problem characterized by a phase change interface whose location varies with time, appears frequently in many areas of applied science such as melting of ice, recrystallization of metals, binary alloy melting and solidification, food preservation, droplet evaporation, oxygen diffusion and particle dissolution. Mathematically, the Stefan problem can be defined as a parabolic partial differential equation with associated initial and boundary conditions which has to be solved in a time-dependent space domain with the phase change interface. Owing to the unknown position of the phase change interface and the non-linear form of the thermal energy balance equation at the interface, the Stefan problem is usually solved by numerical methods [1-5].

In this paper, we consider one-dimensional melting problem in a semi-infinite domain due to the periodically oscillating boundary temperature. The problem can be formulated as:

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial u}{\partial t} \qquad t > 0, \quad 0 < r < R \tag{1}$$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -\mathrm{Ste}\frac{\partial u(R,t)}{\partial r} \qquad t > 0 \tag{2}$$

subject to

$$u(R,t) = 0 \qquad t > 0 \tag{3}$$

$$u(0,t) = 1 + \varepsilon \sin(\omega t) \qquad t > 0, \quad 1 > \varepsilon > -1 \tag{4}$$

$$R(0) = 0 \tag{5}$$

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* Corresponding author; e-mail: qulianghui263@sina.com

where ε is the amplitude of the periodically oscillating boundary temperature, ω – the oscillation frequency, Ste – the Stefan number given by $C\Delta u_{ref}/l$, where C is the specific heat capacity, l – the latent heat, and Δu_{ref} – the reference temperature. In the case of $\varepsilon = 0$, this melting problem corresponds to one-dimensional Stefan problem with time-independent boundary conditions.

Approximate analytical method

Consider the special case of $u(0, t) = 1.0$ corresponding to the Stefan problem with time-independent boundary condition in the half-plane. Then the solution for this phase change process can be given by:

$$R(t) = 2\lambda(\alpha_1 t)^{1/2} \tag{6}$$

$$u(r,t) = 1 - \frac{\operatorname{erf}\dfrac{r(\alpha_1 t)^{-1/2}}{2}}{erf(\lambda)} \tag{7}$$

where erf is the error function, and the value of λ is determined from the transcendental equation:

$$\sqrt{\pi}\lambda\exp(\lambda^2)\operatorname{erf}(\lambda) = \mathrm{Ste} \tag{8}$$

In addition, the mathematical model for the zero latent heat analogue, namely the no-latent-heat melting process, can be expressed as:

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad t > 0 \tag{9}$$

$$u(r, 0) = 0 \tag{10}$$

$$u(0, t) = 1.0 \tag{11}$$

By using Laplace transform, the solution to the above system (9)-(11) can be obtained as:

$$u(r, t) = 1.0 - \operatorname{erf}\frac{r(\alpha t)^{-1/2}}{2} \tag{12}$$

According to the concept of effective thermal diffusivity [6], a penetration depth σ needs to be defined for computing the effective thermal diffusivity. The penetration depth σ is desired to be a small positive number. Thus the position of the phase change temperature, namely $u = 0$, can be approximately expressed as:

$$r_f = 2(\alpha t)^{1/2}\operatorname{erf}^{-1}(1-\sigma) \tag{13}$$

where it is assumed that $\sigma = 0.01$. Let $R = r_f$ in eqs. (6) and (13). By correcting the thermal diffusivity α, the so-called effective thermal diffusivity α_e is obtained as:

$$\alpha_e = \alpha_1\left[\frac{\lambda}{\operatorname{erf}^{-1}(1.0-\sigma)}\right]^2 \tag{14}$$

Then consider the following zero latent heat model corresponding to one-dimensional Stefan problem with periodically oscillating boundary conditions:

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad t > 0 \tag{15}$$

$$u(r,0) = 0 \tag{16}$$

$$u(0,t) = 1.0 + \varepsilon \sin(\omega t) \tag{17}$$

By using Laplace transform, the solution of the above system (15)-(17) is given by [7]:

$$u(r,t) = \operatorname{erfc}\frac{r(\alpha t)^{-1/2}}{2} + 2\pi^{-1/2}\varepsilon \int_{r(\alpha t)^{-1/2}/2}^{\infty} \sin\left[\omega\left(t - \frac{r^2}{4\alpha v^2}\right)\right]\exp(-v^2)\,\mathrm{d}v \tag{18}$$

namely

$$u(r,t) = \operatorname{erfc}\frac{r(\alpha t)^{-1/2}}{2} + \varepsilon\cdot\sin\left[\omega t - r\left(\frac{\omega\alpha^{-1}}{2}\right)^{1/2}\right]\exp\left[-r\left(\frac{\omega\alpha^{-1}}{2}\right)^{1/2}\right] -$$

$$-2\pi^{-1/2}\varepsilon \int_0^{r(\alpha t)^{-1/2}/2} \sin\left[\omega\left(t - \frac{r^2}{4\alpha v^2}\right)\right]\exp(-v^2)\,\mathrm{d}v \tag{19}$$

Let $u(r,t) = \sigma$, $r = R$, and $\alpha = \alpha_e$ in eq. (19). Then it is not difficult to get the desired result as:

$$\operatorname{erfc}\frac{R(\alpha_e t)^{-1/2}}{2} - \sigma + \varepsilon\cdot\sin\left[\omega t - R\left(\frac{\omega\alpha_e^{-1}}{2}\right)^{1/2}\right]\exp\left[-R\left(\frac{\omega\alpha_e^{-1}}{2}\right)^{1/2}\right] -$$

$$-2\pi^{-1/2}\varepsilon \int_0^{R(\alpha_e t)^{-1/2}/2} \sin\left[\omega\left(t - \frac{R^2}{4\alpha_e v^2}\right)\right]\exp(-v^2)\,\mathrm{d}v = 0 \tag{20}$$

Numerical results and discussions

The following numerical experiments are carried out by using the software MATLAB7.1. Table 1 presents the parameter λ obtained by eq. (8) and the effective thermal diffusivity α_e obtained by eq. (14) corresponding to three Stefan numbers Ste = 0.2, 1.0, and 2.0, when $\alpha_1 = 1.0$. Table 1 indicates that the parameter λ and the effective thermal diffusivity α_e obtained by eqs. (8) and (14), respectively, increase as the Stefan number is increased. Furthermore, the results presented in tab. 1 show that the effective thermal diffusivity α_e derived by correcting the thermal diffusivity is much smaller than the original thermal diffusivity α_1.

Table 1. Effective thermal diffusivities for different Stefan numbers for $\alpha_1 = 1.0$

Ste	λ	α_e
0.2	0.3064239	0.0283036
1.0	0.6200626	0.1158956
2.0	0.8006014	0.1932095

It is worthwhile to note that the fourth term in eq. (20) is more complex and corresponds to the instantaneous perturbation caused by the periodically oscillating boundary temperature at $t = 0$. Thus this term is ignored in the present numerical experiments because of its very small effect on the phase change process. Figure 1(a) shows the evolution of the phase change interface as a function of time for three different values of the Stefan number for an oscillation amplitude of 0.5 and frequency of $\pi/2$ and fig. 1(b) shows the temperature distribution in the phase change domain for Ste = 1.0, $\varepsilon = 0.5$, and $\omega = \pi/2$ for the effective thermal diffusivity method. Figures 2(a) and 2(b) show the corresponding numerical results obtained previously using an invariant-space-grid finite difference approach [8].

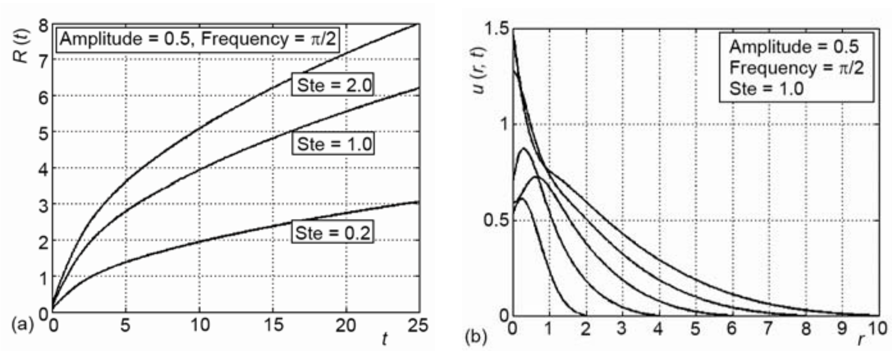

Figure 1. Evolution of the moving interface and temperature distributions for the effective thermal diffusivity method

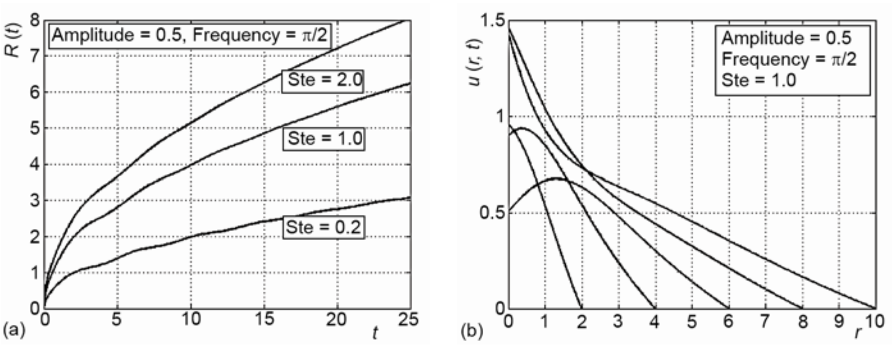

Figure 2. Evolution of the moving interface and temperature distributions for the invariant-space-grid finite different method

For investigating the feasibility and accuracy of the effective thermal diffusivity method, a comparison between the present results and numerical results obtained previously using an invariant-space-grid finite difference approach is made. Table 2 shows that the position of the moving interface obtained with the effective thermal diffusivity method and the invariant-space-grid finite difference method, respectively, at different times. Under the same periodic boundary conditions, the maximum absolute error between the present results and the previous results [8] is 0.0802 and the minimum absolute error is 0.0204 for Ste = 0.2, but the maximum is 0.2866 and the minimum is 0.0437 for Ste = 2.0. Moreover, for three different Stefan numbers Ste = 0.2, 1.0, and 2.0, the absolute errors between the present results and the previous results corresponding to $t = 15.0$, are 0.0386, 0.0441, and 0.0538, respectively, and are 0.0220, 0.0357, and 0.0437 at $t = 24.0$, as can be seen from tab. 2. Consequently, one may conclude that the difference between the present results and the previous results is more pronounced when the Stefan number is large and diminishes as the Stefan number decreases.

In the case of the same Stefan number, the difference becomes smaller and smaller as time increases. Furthermore this conclusion can also be verified by comparing fig. 1(a) with fig. 2(a). Figures 1(b) and 2(b) show that the temperature distribution in the phase change domain for Ste = 1.0, $\varepsilon = 0.5$, and $\omega = \pi/2$, obtained by using the effective thermal diffusivity method and the invariant-space-grid finite difference method, respectively. Good agreement between the present results and the previous results can also be seen from figs. 1(b) and 2(b).

Table 2. Comparison of the position of the moving interface

Ste	t	$R(t)$ (present results)	$R(t)$ (previous results [8])
0.2	3.0	1.0428	1.1230
0.2	6.0	1.5050	1.5653
0.2	10.0	1.9380	1.9815
0.2	15.0	2.3736	2.4122
0.2	20.0	2.7410	2.7614
0.2	24.0	3.0026	3.0246
1.0	3.0	2.1090	2.3299
1.0	6.0	3.0451	3.1002
1.0	10.0	3.9210	3.9610
1.0	15.0	4.8030	4.8471
1.0	20.0	5.5461	5.5882
1.0	24.0	6.0754	6.1111
2.0	3.0	2.7228	3.0094
2.0	6.0	3.9318	3.9832
2.0	10.0	5.0626	5.1240
2.0	15.0	6.2013	6.2551
2.0	20.0	7.1608	7.2093
2.0	24.0	7.8443	7.8880

Conclusions

The effective thermal diffusivity method is used to solve one-dimensional phase change problem with periodically oscillating boundary conditions. Using the effective thermal diffusivity, the approximate analytic method transfers the effect of the latent heat on the phase change process to the effective thermal diffusivity and further transforms the phase change problem into a zero latent heat analogue. Thus an approximate analytic solution showing the functional relation between the location of the phase change interface and time is obtained by using Laplace transform. Furthermore the evolution of the moving interface and the temperature distribution are simulated numerically for three different values of the Stefan number. By comparing the present results with numerical results obtained previously using an invariant-space-grid finite difference approach, the feasibility and effectiveness of the effective thermal diffusivity method are easily demonstrated for solving one-dimensional melting problem with periodic boundary condition. In particular, this approximate analytic method can be used to rapidly evaluate the location of the moving interface and the temperature distribution in the phase change domain in actual application.

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