OSCILLATION BEHAVIOR OF A CLASS OF NEW GENERALIZED EMDEN-FOWLER EQUATIONS

by

Hong-Ying LUO [a*], ***Jun LIU*** [a], ***Xi LIU*** [b], ***and Chun-Yan ZHU*** [a]

[a] College of Mathematics and Information Science, Qujing Normal University, Qujing, China
[b] College of Information Science and Engineering, Yunnan University, Kunming, China

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In this paper, we analyze a class of new generalized Emden-Fowler equations. By using the generalized Riccati transformation and specific analytical skills, new oscillation criteria are obtained which generalize and improve some known results.

Key words: *generalized Emden-Fowler equation, oscillation criteria, generalized Riccati transformation*

Introduction

In this paper, we consider a class of new generalized Emden-Fowler equations with neutral type delays:

$$[r(t)|z'(t)|^{\alpha-1} z'(t)]' + q(t)|x[\sigma(t)]|^{\beta-1} x[\sigma(t)] = 0, \quad t \ge t_0 \tag{1}$$

where $z(t) = x(t) + p(t)x[\tau(\sigma)]$, α and β are constants, $r(t) \in C^1([t_0,\infty],R)$, $p(t)$, $q(t) \in C([t_0,\infty])$, and the followings are satisfied:

(I) $$0 \le p(t) \le 1,\ q(t) \ge 0 .$$

(II) $$r(t) > 0,\ r'(t) \ge 0,\ R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s$$

(III) $$\sigma(t) \in C^1([t_0,\infty],R),\ \sigma(t) > 0,\ \sigma'(t) > 0,\ \sigma(t) \le t,\quad \lim_{t\to\infty}\sigma(t) = \infty$$

When $\alpha = \beta = 1$, $r(t) = 1$, $\sigma(t) = t$, $p(t) = 0$, eq. (1) change to following equation:

$$x'' + q(t)x(t) = 0 \tag{2}$$

Equation (2) has some oscillation behavior [1-3], In 2012, Liu [4] obtained some oscillation criteria of eq. (1). In the last decades, the Emden-Fowler equations have attracted extensive attention for the relevance to nuclear physics and gaseous dynamics in astrophysics, the oscillation behavior of the Emden-Fowler equations are studied by many scholars [5-10].

In this paper, we further study from eq. (1), new oscillation criteria are obtained which generalize and improve some known results, especially the result by Kamenev [2] and Philos [3] becomes a special case of our results.

∗ Corresponding author; e-mail: luohongy1982@163.com

Oscillation behavior

Lemma 1. Assume that $x(t)$ is a positive solution of eq. (1), then we have:

(I) $$z'(t) > 0, \quad z''(t) \le 0$$

(II) $$x(t) \ge [1 - p(t)]z(t)$$

(III) $$\{r(t)[z'(t)]^{\alpha}\}' + Q(t)z^{\beta}[\sigma(t)] \le 0, \text{ where } Q(t) = q(t)\{1 - p[\sigma(t)]\}^{\beta}$$

Proof. (I) Since $x(t)$ is a positive solution of eq. (1) and $x(t) \ge 1$, we have $z(t) > 0$, from eq. (1), we get:

$$[r(t)|z'(t)|^{\alpha-1} z'(t)]' \le 0, \quad t \ge t_0$$

then $r(t)|z'(t)|^{\alpha-1} z'(t)$ is a decreasing function and it is fixed symbol function, that is $r(t)|z'(t)|^{\alpha-1} z'(t) > 0$ or $r(t)|z'(t)|^{\alpha-1} z'(t) < 0.$ Therefore, $z'(t) > 0$ or $z'(t) < 0$.

Suppose that $z'(t) > 0$, Otherwise, if there exists a $t_1 \ge t_0$ such that $z'(t) < 0$ for $t \ge t_1$, then, for some positive number K, we have:

$$r(t)|z'(t)|^{\alpha-1} z'(t) = -r(t)|z'(t)|^{\alpha} = -r(t)[-z'(t)]^{\alpha} \le K, \quad t \ge t_1$$

that is:

$$z'(t) \le -\left[\frac{K}{r(t)}\right]^{1/\alpha}, \quad t \ge t_1$$

Integrating the inequality from t_1 to t:

$$z(t) \le z(t_1) - K^{1/\alpha} \int_{t_1}^{t} \left[\frac{1}{r(s)}\right]^{1/\alpha} \mathrm{d}s$$

letting $t \to \infty$, we get $\lim_{t\to\infty} z(t) = -\infty,$ which contradicts $z(t) > 0$. Thus, we have $z'(t) > 0$.

From eq. (1), we have:

$$[r(t)|z'(t)|^{\alpha-1} z'(t)]' = r'(t)[z'(t)]^{\alpha} + \alpha r(t)[z'(t)]^{\alpha-1} z''(t) \le 0$$

then $z'' \le 0$.

(II) $$x(t) = z(t) - p(t)x[\tau(t)] \ge [1 - p(t)]z(t)$$

(III) $$q(t)|x(\sigma(t))|^{\beta-1} x[\sigma(t)] = q(t)x^{\beta}[\sigma(t)] \ge Q(t)z^{\beta}[\sigma(t)],$$

$$\{r(t)[z'(t)]^{\alpha}\}' + Q(t)z^{\beta}[\sigma(t)] \le 0$$

Lemma 2 [4]. Assume that $\theta > 0$, $A > 0$, $B \in R$, then:

$$Bu - Au^{(\theta+1)/\theta} \le \frac{\theta^{\theta}}{(\theta+1)^{\theta+1}} \frac{B^{\theta+1}}{A^{\theta}}$$

Lemma 3. Assume that $x(t)$ is a positive solution of eq. (1), $\rho(t) \in C^1(I, R^+)$, $\rho'(t) \ge 0$, $I = (t_0, \infty)$, let:

$$w(t)=\rho(t)\frac{r(t)[z'(t)]^{\alpha}}{z^{\beta}[\sigma(t)]}$$

then we have:

(I) $\alpha\ge\beta,\quad w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-A(t)w^{(\beta+1)/\beta}(t),\quad$ where $A(t)=\frac{b^{(\alpha-\beta)/\beta}\beta\sigma'(t)}{[\rho(t)r(t)]^{1/\beta}}$

(II) $\beta\ge\alpha,\quad w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-A(t)[w(t)]^{(\alpha+1)/\alpha}$ where $A(t)=\frac{\alpha\sigma'(t)}{[\rho(t)r(t)]^{1/\alpha}}$

Proof. (I) When $\alpha\ge\beta$, from (III) of Lemma 1, we have:

$$w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{\{r(t)[z'(t)]^{\alpha}\}'}{z^{\beta}[\sigma(t)]}-\rho(t)r(t)[z'(t)]^{\alpha}\frac{\beta z^{\beta-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{z^{2\beta}[\sigma(t)]}\le$$

$$\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-\rho(t)r(t)[z'(t)]^{\alpha}\frac{\beta z'[\sigma(t)]\sigma'(t)}{z^{\beta+1}[\sigma(t)]}\le$$

$$\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-\frac{\beta\sigma'(t)}{[\rho(t)r(t)]^{1/\beta}[z'(t)]^{(\alpha/\beta)-1}}[w(t)]^{(\beta+1)/\beta}$$

from Lemma 1, we have $z''(t)\le 0,$ that is $z'(t)$ is decreasing function and bounded, there exist $b>0$, such that $z'(t)\le 1/b$, we have:

$$w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-\frac{b^{(\alpha-\beta)/\beta}\beta\sigma'(t)}{[\rho(t)r(t)]^{1/\beta}}[w(t)]^{(\beta+1)/\beta}=$$

$$=\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-A(t)[w(t)]^{(\beta+1)/\beta}$$

(II) When $\beta\le\alpha$, from (III) of Lemma 1, we have:

$$w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-\frac{\beta z'[\sigma(t)]\sigma'(t)}{[\rho(t)r(t)]^{1/\alpha}z'(t)[z(t)]^{1-(\beta/\alpha)}}\left\{\frac{\rho(t)r(t)[z'(t)]^{\alpha}}{z^{\beta}[\sigma(t)]}\right\}^{(\alpha+1)/\alpha}$$

from Lemma 1, we have $z''(t)\le 0,$ that is $z'(t)$ is decreasing function, since $t\ge\sigma(t)$, then $z'(t)\le z'[\sigma(t)],$ we have:

$$w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-\frac{\beta\sigma'(t)}{[\rho(t)r(t)]^{1/\alpha}[z(t)]^{1-(\beta/\alpha)}}[w(t)]^{(\alpha+1)/\alpha}\le$$

$$\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q(t)-A(t)[w(t)]^{(\alpha+1)/\alpha}$$

Theorem 1. Assume that $\beta\ge\alpha$:

$$\overline{\lim_{t\to\infty}}\frac{1}{t^{n}}\int_{t_0}^{t}\left((t-s)^{n}\left\{\rho(s)Q(s)-\left[\frac{\rho'(s)}{\alpha+1}\right]^{\alpha+1}\frac{r(s)}{[\rho(s)\sigma'(s)]^{\alpha}}\right\}\right)\mathrm{d}s=\infty$$

where $Q(t)=q(t)\{1-p[\sigma(t)]\}^{\beta}$, then eq. (1) is oscillatory.

Proof. Suppose that eq. (1) has a non-increasing positive solution $x(t)$, from Lemma 3 and Lemma 2, when $t > t_1$, we have:

$$w'(t) \le -\rho(t)Q(t) + \left[\frac{\rho'(t)}{\alpha+1}\right]^{\alpha+1} \frac{r(t)}{[\rho(t)\sigma'(t)]^{\alpha}} = -\varphi(t)$$

that is $\varphi(t) \le -w'(t)$, we get:

$$\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n \varphi(s)\mathrm{d}s \le \frac{1}{t^n}\int_{t_1}^{t}[-(t-s)^n w'(s)\mathrm{d}s = \frac{1}{t^n}(t-t_1)^n w(t_1) -$$

$$-\frac{1}{t^n}\int_{t_1}^{t} n(t-s)^{n-1} w(s)\mathrm{d}s \le \left(1-\frac{t_1}{t}\right)^n w(t_1)$$

Since $\overline{\lim_{t\to\infty}}[1-(t_1/t)]^n w(t_1) = w(t_1) < \infty,$ which contradicts conditions, then eq. (1) is oscillatory.

Let $\rho(t) = 1$, have:

$$\overline{\lim_{t\to\infty}}\frac{1}{t^n}\int_{t_0}^{t} (t-s)^n q(s)\mathrm{d}s = \infty,$$

the ref. [2] have spread.

Theorem 2. Assume that $\alpha \ge \beta$:

$$\overline{\lim_{t\to\infty}}\frac{1}{t^n}\int_{t_0}^{t}\left\{(t-s)^n\left[\rho(s)Q(s) - \left[\frac{\rho'(s)}{\beta+1}\right]^{\beta+1}\frac{r(s)}{[\rho(s)\sigma'(s)]^{\beta} b^{\alpha-\beta}}\right]\right\}\mathrm{d}s = \infty$$

where $Q(t) = q(t)\{1 - p[\sigma(t)]\}^{\beta}$, then eq. (1) is oscillatory.

Now, we consider set $D = [(t, s)\colon t \ge s \ge t_0]$, if $H(t, s) \in C(D, R)$ and satisfy follows conditions, then remember to $H(t, s) \in \Omega$:

(I) $\quad H(t, t) = 0,\ t \ge t_0;\ H(t, s) > 0,\ t > s \ge t_0$

(II) $\quad \dfrac{\partial H(t,s)}{\partial s} \le 0,$ and is continuous on D.

(III) $\quad \exists \rho(t) \in C^1, \quad \rho'(t) \ge 0, \quad h(t,s) \in C(D,R),$ such that:

$$\beta \ge \alpha, \quad \frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = -h(t,s)H^{\alpha/(\alpha+1)}(t,s)$$

$$\alpha \ge \beta, \quad \frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = -h(t,s)H^{\beta/(\beta+1)}(t,s)$$

Theorem 3. Assume that $\beta \ge \alpha$, $H(t, s) \in \Omega$:

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)\rho(s)Q(s) - \left[\frac{|h(t,s)|}{\alpha+1}\right]^{\alpha+1}\frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}}\right\}\mathrm{d}s = \infty$$

where $Q(t) = q(t)\{1 - p[\sigma(t)]\}^{\beta}$, then eq. (1) is oscillatory.

Proof. Suppose that eq. (1) has a positive solution $x(t)$, from Lemma 2-3, when $t > t_1 > t_0$, we have:

$$\rho(t)Q(t) \le -w'(t) + \frac{\rho'(t)}{\rho(t)} w(t) - A(t)[w(t)]^{(\alpha+1)/\alpha}$$

$$\int_{t_1}^{t} \rho(s)Q(s)\mathrm{d}s \le \int_{t_1}^{t} \left[-w'(s) + \frac{\rho'(s)}{\rho(s)} w(s) - A(s) w^{(\alpha+1)/\alpha}(s) \right] \mathrm{d}s \cdot$$

$$\cdot \int_{t_1}^{t} H(t,s)\rho(s)Q(s)\mathrm{d}s \le \int_{t_1}^{t} H(t,s) \left[-w'(s) + \frac{\rho'(s)}{\rho(s)} w(s) - A(s) w^{(\alpha+1)/\alpha}(s) \right] \mathrm{d}s \le$$

$$\le H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[\frac{|h(t,s)|}{\alpha+1} \right]^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \mathrm{d}s \cdot$$

$$\cdot \int_{t_1}^{t} \left\{ H(t,s)\rho(s)Q(s) - \left[\frac{|h(t,s)|}{\alpha+1} \right]^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \right\} \mathrm{d}s \le H(t,t_1)w(t_1) \le H(t,t_0)w(t_1)$$

$$\int_{t_0}^{t} \left\{ H(t,s)\rho(s)Q(s) - \left(\frac{|h(t,s)|}{\alpha+1} \right)^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \right\} \mathrm{d}s =$$

$$= \int_{t_0}^{t_1} \left\{ H(t,s)\rho(s)Q(s) - \left(\frac{|h(t,s)|}{\alpha+1} \right)^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \right\} \mathrm{d}s +$$

$$+ \int_{t_1}^{t} \left\{ H(t,s)\rho(s)Q(s) - \left(\frac{|h(t,s)|}{\alpha+1} \right)^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \right\} \mathrm{d}s \le$$

$$\le H(t,t_0) \left[\int_{t_0}^{t_1} \rho(s)Q(s)\mathrm{d}s + w(t_1) \right].$$

That is, we have:

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)Q(s) - \left[\frac{|h(t,s)|}{\alpha+1} \right]^{\alpha+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\alpha}} \right\} \mathrm{d}s \le \int_{t_0}^{t_1} \rho(s)Q(s)\mathrm{d}s + w(t_1)$$

which contradicts conditions, then eq. (1) is oscillatory.

Let $\rho(t) = 1$, $p(t) = 0$, $\sigma(t) = t$, $r(t) = 1$, $\alpha = 1$, we have:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)q(s) - \frac{1}{4} h^2(t,s)]\mathrm{d}s = \infty$$

the ref. [3] have spread.

Corollary 1. Assume that $\beta \geq \alpha$, $H(t, s) \in \Omega$:

$$\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \left|h(t,s)\right|^{\alpha+1} \right] \mathrm{d}s = \infty$$

then eq. (1) is oscillatory.

Theorem 4. Assume that $\alpha \geq \beta$, $H(t, s) \in \Omega$:

$$\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)Q(s) - \left[\frac{\left|h(t,s)\right|}{\beta+1}\right]^{\beta+1} \frac{\rho(s)r(s)}{[\sigma'(s)]^{\beta}} \frac{1}{b^{\alpha-\beta}} \right\} \mathrm{d}s = \infty$$

where $Q(t) = q(t)\{1 - p[\sigma(t)]\}^{\beta}$, then eq. (1) is oscillatory.

Corollary 2. Assume that $\alpha \geq \beta$, $H(t, s) \in \Omega$:

$$\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \left(\frac{1}{\beta+1}\right)^{\beta+1} \left|h(t,s)\right|^{\beta+1} \frac{1}{b^{\alpha-\beta}} \right] \mathrm{d}s = \infty .$$

then eq. (1) is oscillatory.

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