A TODA LATTICE HIERARCHY WITH VARIABLE COEFFICIENTS AND ITS MULTI-WAVE SOLUTIONS

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Starting from the Toda spectral problem, a new Toda lattice hierarchy of isospectral equations with variable coefficients is constructed through the discrete zero curvature equation. In order to solve one special case of the derived Toda lattice hierarchy, a series of appropriate transformations are utilized. As a result, a new uniform formula of N-wave solutions is obtained.

Key words: *Toda lattice hierarchy, isospectral equation, zero curvature equation, multi-wave solution*

Introduction

It is well known that non-linear physical phenomena are often related to some non-linear evolution equations (NLEE). When the inhomogeneities of media and non-uniformities of boundaries are taken into account, the variable-coefficient NLEE could describe more realistic physical phenomena than their constant-coefficient counterparts [1]. Constructing such types of NLEE and obtaining their exact solutions often play an important role in helping us understand these phenomena. Since Ablowitz successfully solved the isospectral Toda equation, there has been various works for the Toda hierarchy, such as those in [2-4], but almost of these works focus on constant-coefficient hierarchy. In this paper, starting from the Toda spectral problem, we shall construct a new Toda lattice hierarchy with variable coefficients and then construct its multi-wave solutions.

A Toda lattice hierarchy with variable coefficients

Let E be the shift operator defined as $Eu_n = u_{n+1}$, $E^{-1}u_n = u_{n-1}$. If u_n goes to zero as $|n|\to\infty$, we also define the inverse operator of $E - E^{-1}$ as:

$$(E-E^{-1})^{-1}u_n = -\sum_{m=n}^{\infty} u_{2m-n+1}, \quad (E-E^{-1})^{-1}u_n = \sum_{m=-\infty}^{n} u_{2m-n-1} \tag{1}$$

With the preparation, we consider the spectral problem:

$$E\varphi_n = M\varphi_n, \quad M = \begin{pmatrix} 0 & 1 \\ -\alpha(t)u_n & \lambda-\beta(t)v_n \end{pmatrix}, \quad \varphi_n = \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix} \tag{2}$$

and the time evolution:

$$\varphi_{n,t} = N\varphi, \quad N = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \tag{3}$$

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where $u_n = u_n(t)$ and $v_n = v_n(t)$ are potentials, and λ is the spectral parameter. We assume that u_n and v_n are smooth functions of t and (u_n, v_n) goes to (1,0) rapidly as $|n|\to\infty$. The compatibility condition of eqs. (2) and (3) reads $M_t = (EN)M - MN$, which gives:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1\begin{pmatrix} D_n \\ B_n \end{pmatrix} - \lambda L_2\begin{pmatrix} D_n \\ B_n \end{pmatrix} - \begin{bmatrix} \alpha'(t)\alpha^{-1}(t)u_n \\ \beta'(t)\beta^{-1}(t)v_n \end{bmatrix} + \lambda_t\begin{bmatrix} 0 \\ \beta^{-1}(t) \end{bmatrix} \tag{4}$$

where

$$L_1 = \begin{bmatrix} u_n(E-E^{-1}) & \beta(t)u_n(E-1)v_{n-1} \\ v_n(E-1) & \alpha(t)\beta^{-1}(t)(Eu_nE-u_n) \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & u_n(E-1) \\ \beta^{-1}(t)(E-1) & 0 \end{bmatrix}$$

Taking $\lambda_t = 0$ and expanding $(D_n, B_n)^T$ as:

$$\begin{pmatrix} D_n \\ B_n \end{pmatrix} = \sum_{j=0}^{k}\begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}\lambda^{k-j}, \quad D_n = B_n = \lambda^k, \quad (u_n = 1, v_n = 0) \tag{5}$$

and then comparing the coefficients of the same power of λ, we can obtain:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1\begin{pmatrix} d_{n,k} \\ b_{n,k} \end{pmatrix}, \quad L_2\begin{pmatrix} d_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = L_1\begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}, \quad (j = 0,1,...,k-1), \quad L_2\begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix} = 0 \tag{6}$$

A direct computation on the second equation of eq. (6) gives $d_{n.0} = b_{n.0} = 1$ which satisfies the last equation of eq. (6). If further taking:

$$L = L_1L_2^{-1} = \begin{bmatrix} \beta(t)u_n(E-1)v_{n-1}(E-1)^{-1}u_n^{-1} & \beta(t)u_n(1+E^{-1}) \\ \alpha(t)\beta^{-1}(t)(Eu_nE-u_n)(E-1)^{-1}u_n^{-1} & \beta(t)v_n \end{bmatrix} \tag{7}$$

from the first two equations of eq. (6) we obtain a new Toda lattice hierarchy in the form:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k\begin{bmatrix} \beta(t)u_n(v_n - v_{n-1}) \\ \alpha(t)\beta^{-1}(t)(u_{n+1}-u_n) \end{bmatrix} - \begin{bmatrix} \alpha'(t)\alpha^{-1}(t)u_n \\ \beta'(t)\beta^{-1}(t)v_n \end{bmatrix}, \quad (k = 0,1,2,\cdots) \tag{8}$$

Multi-wave solutions

If set $k = 0$, from eq. (8) we have:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \begin{bmatrix} \beta(t)u_n(v_n - v_{n-1}) \\ \alpha(t)\beta^{-1}(t)(u_{n+1}-u_n) \end{bmatrix} - \begin{bmatrix} \alpha'(t)\alpha(t)u_n \\ \beta'(t)\beta^{-1}(t)v_n \end{bmatrix} \tag{9}$$

In this section, we would like to construct multi-wave solutions of eq. (9). To begin with, we take a transformation in the form:

$$u_n = \alpha^{-1}(t)\mathrm{e}^{x_{n-1}-x_n}, \quad v_n = -\beta^{-1}(t)x_{n,t} \tag{10}$$

then eq. (9) is reduced to:

$$x_{n,tt} = \mathrm{e}^{x_{n-1}-x_n} - \mathrm{e}^{x_n - x_{n+1}} \tag{11}$$

which is the known constant-coefficient Toda lattice equation [5].

If set $y_n = x_n - x_{n+1}$ and hence $y_{n-1} = x_{n-1} - x_n$, from eq. (11) we have:

$$x_{n,tt} = \mathrm{e}^{y_{n-1}} - \mathrm{e}^{y_n}, \quad x_{n+1,tt} = \mathrm{e}^{y_n} - \mathrm{e}^{y_{n+1}} \tag{12}$$

then eq. (11) becomes:

$$y_{n,tt} = \mathrm{e}^{y_{n-1}} - 2\mathrm{e}^{y_n} + \mathrm{e}^{y_{n+1}} \tag{13}$$

Supposing $z_{n,t} = \mathrm{e}^{y_n} - 1$, then we have:

$$\mathrm{e}^{y_n} = z_{n,t} + 1, \quad \mathrm{e}^{y_{n-1}} = z_{n-1,t} + 1, \quad \mathrm{e}^{y_{n+1}} = z_{n+1,t} + 1 \tag{14}$$

Substituting eq. (14) into eq. (13) yields:

$$y_{n,tt} = z_{n-1,t} - 2z_{n,t} + z_{n+1,t} \tag{15}$$

then integrating eq. (15) with respect to t once and setting the integration constant to zero, we obtain:

$$y_{n,t} = z_{n-1} - 2z_n + z_{n+1} \tag{16}$$

We further differentiate the first equation of eq. (14) with respect to t once, then $z_{n,tt} = y_{n,t}\mathrm{e}^{y_n}$, from which the following equation is obtained by use of eqs. (14) and (16):

$$z_{n,tt} = (z_{n,t} + 1)(z_{n-1} - 2z_n + z_{n+1}) \tag{17}$$

Employing the generalized Exp-function method [6], we can construct a uniform formula of N-wave solution of eq. (17):

$$z_n = \left[\ln\left(\sum_{\mu=0,1}\prod_{i=1}^{N} b_i^{\mu_i}\mathrm{e}^{\Sigma_{i=1}^{N}\mu_i\xi_i + \Sigma_{1\le i<j\le N}\mu_i\mu_j B_{ij}}\right)\right]_t, \quad \xi_i = k_i n + 2\sinh\frac{k_i}{2}t + w_i \tag{18}$$

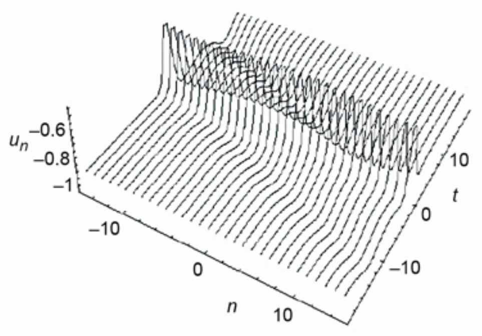
where the summation $\Sigma_{u=0.1}$ refers all combination of each μ_1 = 0,1 for i = 1, 2, …, N, $i = 1, 2, \ldots, N$, $\mathrm{e}^{B_{ij}} = (k_i - k_j)^2/(k_i + k_j)^2$ ($i < j$; j = 1, 2, …, N), b_i, k_i, and w_i are arbitrary constants.

Finally, with the help of eqs. (10), (12), (14), and (18) we obtain a uniform formula of N-wave solutions of eq. (9):

$$u_n = \alpha^{-1}(t)(z_{n-1,t} + 1), \quad v_n = -\beta^{-1}(t)\int(z_{n-1,t} - z_{n,t})\,\mathrm{d}t \tag{19}$$

where z_{n-1} and z_n are determined by eq. (18).

To the best of our knowledge, the N-wave solutions (19) have not been reported in literature. In fig. 1, a pair of double-wave structures of solutions (19) are shown.



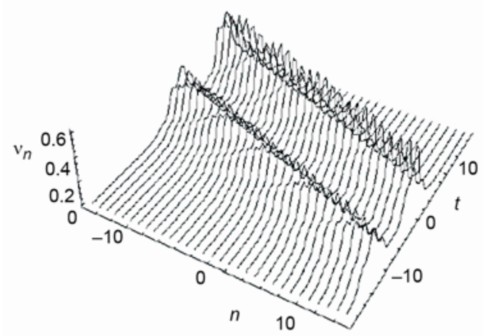


Figure 1. Double-wave structures of solutions (19) with parameters $N = 2$, $\alpha(t) = -1 - \text{sech}^2(-t)$, $\beta(t) = 1 + \sin^2(t)$, $b_1 = 0.2$, $b_2 = 1$, $k_1 = 0.3$, $k_2 = -0.36$, $w_1 = 0$, $w_2 = 0$

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