APPROXIMATE SOLUTIONS OF FRACTIONAL NON-LINEAR EVOLUTION EQUATIONS

by

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A novel method which is based on variational iteration method, Laplace transform, and homotopy perturbation method is proposed, and this new method is applied to obtain the approximate solution of the fractional non-linear Boussinesq-type equation. The fractional Lagrange multiplier is accurately determined by the Laplace transform and the non-linear term can be easily handled by He's polynomials. The result demonstrates accuracy and fast convergence of this new algorithm.

Key words: *variational iteration method, Laplace transform, homotopy perturbation method, fractional equation, non-linear equation, Caputo derivative*

Introduction

The fractional non-linear equations are successfully applied to several situations due to its wide applications in engineering such as frequency-dependent damping behavior of materials, viscoelasticity, diffusion processes, *etc.* [1]. With the development of non-linear sciences, many analytical and numerical techniques [2, 3] have been developed by various scientists. But these fractional differential equations are difficult to get their exact solutions [4, 5]. So, the numerical methods have largely been used to solve these equations. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work. Recently, some improved homotopy perturbation methods [6] and improved variational iteration method [7] have been used by many researches.

Wu [8] applies Laplace transform to the variational iteration method (VIM). Motivated and inspired by the on-going research in this field, we give a novel modified method which is based on variational iteration method, Laplace transform and homotopy perturbation method. The fractional Lagrange multiplier is accurately determined by the Laplace transform and the non-linear can be easily handled by use of He's polynomials.

Preliminaries

Definition 1: The Caputo time-fractional derivative operator of order $\alpha > 0$ is defined:

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-a-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} \mathrm{d}\tau, \quad m = [\alpha] + 1, \quad m \in N \tag{1} $$

where $\Gamma(-)$ denotes the gamma function.

Definition 2: Laplace transform of ${}_0^C D_t^\alpha u$ is given as:

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$$L\left[{}_0^C D_t^\alpha u(x,t)\right] = s^\alpha U(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \le m \tag{2}$$

where $U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)dt$. And further information about fractional derivatives and its properties can be found in [1].

Description of the method

Let us consider the time fractional equation as:

$${}_0^C D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{3}$$

where $u^{(k)}(x,0^+) = a_k, k = 0,\cdots, m-1,\ m = [\alpha]+1,\ g(x,t)$ is the source term, N – the general non-linear operator, and R – the linear operator. Now, we taking the above Laplace transform to both sides of eq. (3), the iteration formula can be constructed as:

$$U_{n+1} = U_n + \lambda(s)\left\{s^\alpha U_n - \sum_{k=0}^{m-1} U^{(k)}(x,0^+) s^{\alpha-1-k} + L\left\{R[u_n] + N[u_n] - g(x,t)\right\}\right\} \tag{4}$$

considering $L\{R[u_n(x,t)] + N[u_n(x,t)]\}$ as restricted terms, one can derive a Lagrange multiplier as $\lambda = -1/s^\alpha$, taking the inverse-Laplace transform L^{-1}, the eq. (4) can be explicitly given as:

$$u_{n+1} = u_0 - L^{-1}\left[\frac{1}{s^\alpha}\left(L\ \{R[u_n(x,t)] + N[u_n(x,t)]\}\right)\right] \tag{5}$$

where u_0 is an initial approximation of eq. (3), and:

$$u_0 = L^{-1}\left[\sum_{k=0}^{m-1} u^{(k)}(x,0^+) s^{\alpha-1-k}\right] + L^{-1}\left\{\frac{1}{s^\alpha} L[g(x,t)]\right\} \tag{6}$$

How to deal with the non-linear terms, we will give the improved method which based on the above method [8]. In the homotopy method, the basic assumption is that the solutions can be written as a power series in p: $u(x,t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots$, and the non--linear term can be decomposed as $Nu(x,t) = \sum_{n=0}^\infty p^n H_n(u)$ where $p \in [0,1]$ is an embedding parameter. $H_n(u)$ is He's polynomials [6, 9] can be generated by:

$$H_n(u_0,\cdots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\sum_{i=0}^n p^i u_i\right)_{p=0}, \quad n = 0,1,2,\cdots \tag{7}$$

The variational homotopy perturbation method is obtained by the elegant coupling of correction function (5) with He's polynomials and is given by:

$$\sum_{n=0}^\infty p^n u_n(x,t) = u_0 - p\left(L^{-1}\left\{\frac{1}{s^\alpha} L\left[R\sum_{n=0}^\infty p^n u_n(x,t)\right] + \frac{1}{s^\alpha} L\left[\sum_{n=0}^\infty p^n H_n(u)\right]\right\}\right) \tag{8}$$

where u_0 represents the term arising form the source term and the initial conditions. Equating the terms with identical powers in p, we obtain the approximations:

$$p^0: u_0(x,t) = u(x,0) + u'(x,0)t + \cdots + \frac{u^{(m-1)}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left\{\frac{1}{s^\alpha} L[g(x,t)]\right\} \tag{9}$$

$$p^1 : u_1(x,t) = -L\left\{\frac{1}{s^\alpha}L[Ru_0(x,t)] + \frac{1}{s^\alpha}L[H_0(u)]\right\} \tag{10}$$

etc.

The best approximation for the solution is $u(x,t) = \sum_{n=0}^{\infty} u_n$.

Applications

Consider the following non-linear time fractional Boussinesq-type equation [10]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - u_{xx} - (u^2)_{xx} + (uu_{xx})_{xx} = 0 \tag{11}$$

where $u(x,0) = -(c^2-1)[\cosh(x)-1]$, $u_t(x,0) = (c^2-1)\sinh(x)$, where $1 < \alpha < 2$ is the time fractional derivative defined in Caputo sense. After taking the Laplace transform on both sides of eq. (11):

$$U_{n+1} = U_n + \lambda(s)\left\{s^\alpha U_n - s^{\alpha-1}u(x,0) - s^{\alpha-2}u_t(x,0) + L[-u_{nxx} - (u_n{}^2)_{xx} + (u_n u_{nxx})_{xx}]\right\} \tag{12}$$

Lagrange multiplier $\lambda(s) = -1/s^\alpha$, and with the inverse Laplace transform, one derive:

$$u_{n+1}(x,t) = u_0 + L^{-1}\left(\frac{1}{s^\alpha}\left\{L[-u_{nxx} - (u_n{}^2)_{xx} + (u_n u_{nxx})_{xx}]\right\}\right) \tag{13}$$

where $u_0 = -(c^2-1)[\cosh(x) - 1 - \sinh(x)t]$. Applying this new modified method, one has:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - p\left[L^{-1}\left(\frac{1}{s^\alpha}\left\{-L\left[\sum_{n=0}^{\infty} p^n u_{nxx} - \sum p^n H_{n1}(u) + \sum_{n=0}^{\infty} p^n H_{n2}(u)\right]\right\}\right)\right] \tag{14}$$

where $H_{n1}(u)$ and $H_{n2}(u)$ are He's polynomials that represents non-linear term $(u^2)_{xx}$ and $(uu_{xx})_{xx}$. We have a few polynomials for $(u^2)_{xx}$ and $(uu_{xx})_{xx}$, which given by $H_{10}(u) = (u_0u_0)_{xx}$, $H_{11}(u) = (2u_0u_1)_{xx}$, $H_{12}(u) = (2u_0u_2 + u_1u_1)_{xx}$, …, $H_{20}(u) = (u_0u_{0xx})_{xx}$, $H_{21}(u) = (u_0u_{1xx} + u_1u_{0xx})_{xx}$, $H_{22}(u) = (u_0u_{2xx} + u_2u_{0xx} + u_1u_{1xx})_{xx}$, … . Comparing the coefficient powers of p, one has:

$$u_0(x,t) = -(c^2-1)[\cosh(x)-1] + (c^2-1)\sinh(x)t \tag{15}$$

$$u_1 = -c^2(c^2-1)\cosh(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + c^3(c^2-1)\sinh(x)\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} \tag{16}$$

$$u_2 = -c^4(c^2-1)\cosh(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c^5(c^2-1)\sinh(x)\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} \tag{17}$$

etc.

the solution of eq. (11) is given:

$$u = -(c^2-1)\left\{\cosh(x)\left[1 + \frac{c^2t^\alpha}{\Gamma(1+\alpha)} + \frac{c^4t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots\right] - 1\right\} +$$

$$+(c^2-1)\sinh(x)\left\{ct+\frac{c^3t^{\alpha+1}}{\Gamma(2+\alpha)}+\frac{c^5t^{2\alpha+1}}{\Gamma(2+2\alpha)}+...\right\} \tag{18}$$

if we take $\alpha = 2$, on has:

$$u=-(c^2-1)\left[\cosh(x)\left(1+\frac{c^2t^2}{2!}+\frac{c^4t^4}{4!}+\cdots\right)-1\right]+(c^2-1)\sinh(x)\left(ct+\frac{c^3t^3}{3!}+\frac{c^5t^5}{5!}+\cdots\right) \tag{19}$$

in close form is $u(x,t)=-2(c^2-1)\sinh^2[(1/2)(x-ct)]$ which is exactly the same solution obtained in [10] using sine-cosine method.

Conclusions

In this paper, a new modified method which is based on variational iteration method, Laplace transform, and homotopy perturbation method is considered. The fractional Lagrange multiplier is accurately determined without tedious calculation from Laplace transform and the non-linear can be easily handled using the He's polynomials. Fractional non-linear non--homogeneous Boussinessq-type equation is analytically solved as example. And the result shows that this method is a powerful and reliable method for finding the solutions of the fractional non-linear evolution equations.

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