OSCILLATION CRITERIA FOR HALF-LINEAR FUNCTION DIFFERENTIAL EQUATIONS WITH DAMPING

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In this paper, a class of half-linear functional differential equations with damping are studied. By using the generalized Riccati transformation and integral average skills, new oscillation criteria are obtained which generalize and improve some known results.

Key words: *differential equation, oscillation criteria, half-linear*

Introduction

A functional differential equation can describe a generalized heat problem [1]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{r(t)\phi[x^{(n-1)}(t)]\}+p(t)\phi[x^{(n-1)}(t)]+q(t)\phi\{x[g(t)]\}=0, \quad t\ge t_0 \tag{1}$$

In this paper, we mainly study the oscillation criteria of a half-linear functional differential equ ation with damping:

$$\{r(t)\varphi_\alpha[z^{(n-1)}(t)]\}'+p(t)\varphi_\alpha[z^{(n-1)}(t)]+q(t)\varphi_\beta\{x[\sigma(t)]\}=0, \quad t\ge t_0 \tag{2}$$

where $\varphi_\alpha(s)=|s|^{\alpha-1}s,\ \varphi_\beta(s)=|s|^{\beta-1}s,\ \alpha$ and β are positive constants, and n is even number,

$$z(t) = x(t) + f(t)x[\tau(t)] \tag{3}$$

and the following conditions are assumed to hold:

$$(H_1) \quad r(t)\in C^1(I,R), I=[t_0,\infty], r(t)>0, r'(t)\ge 0$$

$$(H_2) \quad p(t), q(t), f(t)\in C(I,R), p(t)\ge 0, q(t)>0, 0\le f(t)\le 1$$

$$(H_3) \quad \sigma(t)\in C^1(I,R^+), 0<\sigma(t)\le t, \lim_{t\to\infty} g(t)=\infty, \sigma'(t)>0$$

$$(H_4) \quad \tau(t)\in C^1(I,R), \tau(t)\le t, \lim_{t\to\infty}\tau(t)=\infty$$

Obviously, eq. (1) is a special case of the eq. (2). When $f(t) = 0$, eq. (2) is n-order half-linear functional differential equation with damping:

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$$\{r(t)\varphi_\alpha[x^{(n-1)}(t)]\}' + p(t)\varphi_\alpha[x^{(n-1)}(t)] + q(t)\varphi_\beta\{x[\sigma(t)]\} = 0, \quad t \ge t_0 \tag{4}$$

when $p(t) = 0$, eq. (4) is non-damping differential equation.

Using the generalized Riccati transformation and integral average skills, we obtain new oscillation criteria of eq. (2) which generalize and improve some known results.

Oscillation criteria

Consider the following equation:

$$x''(t) + q(t)x(t) = 0, \quad q(t) > 0 \tag{5}$$

Theorem 1. Let $w(t) = x'(t)/x(t) > 0$, then $w'(t) = -q(t) - w^2(t)$, and eq. (5) is oscillatory.

Proof. Suppose that eq. (5) has a non-oscillatory positive solution $x(t) > 0$, then $x''(t) < 0$, that is $x'(t)$ is decreasing function, therefore $x'(t) > 0$ or $x'(t) < 0$. Suppose $x'(t) < 0$, there exists a $t_1 \ge t_0$ such that $x'(t) < 0$, we have $x'(t) \le x'(t_1) < 0$, $t > t_1$. We get $x(t) \le x(t_1) + x'(t_1)\cdot(t - t_1)$, letting $t \to \infty$, we have $x(t) = -\infty$, which contradicts $x(t) > 0$, therefore $x'(t) > 0$. since $w(t) = x'(t)/x(t) > 0$, we have $w'(t) = -q(t) - w^2(t) < -q(t)$, that is, we get $w(t) \le w(t_1) - \int_{t_1}^{t} q(s)\mathrm{d}s$, letting $t \to \infty$, we have $w(t) = -\infty$, which contradicts $w(t) > 0$, then eq. (5) has not a positive solution. Similarly, then eq. (5) has not a negative solution. Therefore, eq. (5) is oscillatory.

Lemma [2]. Assume that:

$$\theta > 0, \quad A > 0, \quad B \in R, \quad \text{then} \quad Bu - Au^{\frac{\theta+1}{\theta}} \le \frac{\theta^\theta}{(\theta+1)^{\theta+1}} \frac{B^{\theta+1}}{A^\theta}$$

Theorem 2. Assume that $(H_1) \sim (H_4)$ and and $\beta \ge \alpha > 0$ hold, if for each $t \ge t_0$, there exist a function $\rho(t) \in C^1[I, (0, \infty)]$, $\rho(t) \ge 0$ such that:

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{r(s)} \int_{s}^{\infty} Q(u)\mathrm{d}u \right]^{\frac{1}{\alpha}} \mathrm{d}s = \infty, \quad \overline{\lim_{t\to\infty}} \int_{t_0}^{t} \left\{ \rho(s)Q(s) - \left[\frac{\rho'(s)}{\alpha+1}\right]^{\alpha+1} \frac{r(s)}{[\rho(s)\sigma'(s)]^\alpha} \right\} \mathrm{d}s = \infty \tag{6}$$

where $Q(t) = q(t)\{1 - p[\sigma(t)]\}^\beta$, then eq. (2) is oscillatory.

Proof. Suppose that eq. (2) has a positive solution $x(t)$, then have:

$$\{r(t)\varphi_\alpha[z^{(n-1)}(t)]\}' = \left[r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1} z^{(n-1)}(t) \right]' \le 0, \quad t \ge t_0 \tag{7}$$

Therefore $r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1} z^{(n-1)}(t)$ is a non-increasing function and $z'(t)$ is eventually of one sign. That is $z'(t) > 0$ or $z'(t) \le 0$. Otherwise, if there exists a $t_1 \ge t_0$ such that $z'(t) < 0$ for $t \ge t_1$, then from (7) , for some positive K, we have:

$$r(t)\left|z^{(n-1)}(t)\right|^{\alpha-1} z^{(n-1)}(t) = -r(t)\left|z^{(n-1)}(t)\right|^{\alpha} = -r(t)[-z^{(n-1)}(t)]^\alpha \le K, \quad t \ge t_1$$

that is:

$$z^{(n-1)}(t) \le -\left[\frac{K}{r(t)}\right]^{\frac{1}{\alpha}}, \quad t \ge t_1$$

Integrating the inequality from t_1 to t:

$$z^{(n-2)}(t) \le z^{(n-2)}(t_1) - K^{\frac{1}{\alpha}} \int_{t_1}^{t} \left[\frac{1}{r(t)}\right]^{\frac{1}{\alpha}} \mathrm{d}s$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} z^{(n-2)}(t) = -\infty$. Thus, we have $z^{(n-1)}(t) > 0$.

$$x(t) = z(t) - f(t)x[\tau(t)] \ge [1 - f(t)]z(t)$$

$$p(t)\varphi_\alpha[z^{(n-1)}(t)] = p(t)\left|z^{(n-1)}(t)\right|^{\alpha-1} z^{(n-1)}(t) \ge 0$$

$$q(t)\varphi_\beta\{x[\sigma(t)]\} = q(t)\left|x[\sigma(t)]\right|^{\beta-1} x[\sigma(t)] \ge q(t)\{1 - p[\sigma(t)]\}^\beta z^\beta[\sigma(t)] = Q(t)z^\beta[\sigma(t)]$$

Therefore, we have:

$$\{r(t)[z^{(n-1)}(t)]^\alpha\}' + Q(t)z^\beta[\sigma(t)] \le 0, \quad Q(t) \le -\frac{\{r(t)[z^{(n-1)}(t)]^\alpha\}'}{z^\beta[\sigma(t)]}$$

Define:

$$w(t) = \rho(t)\frac{r(t)[z^{(n-1)}(t)]^\alpha}{z^\beta[\sigma(t)]} \tag{8}$$

Suppose $w(t) > 0$, we have:

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{\{r(t)[z^{(n-1)}(t)]^\alpha\}'}{z^\beta[\sigma(t)]} - \rho(t)r(t)[z^{(n-1)}(t)]^\alpha \frac{\beta z^{\beta-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{z^{2\beta}[\sigma(t)]} \le$$

$$\le -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\sigma'(t)}{[\rho(t)r(t)]^{\frac{1}{\alpha}}}[w(t)]^{\frac{\alpha+1}{\alpha}} \tag{9}$$

From Lemma and (9), we have:

$$w'(t) \le -\rho(t)Q(t) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\left[\frac{\rho'(t)}{\rho(t)}\right]^{\alpha+1} \frac{\rho(t)r(t)}{\alpha^\alpha[\sigma'(t)]^\alpha}$$

$$w(t) \le w(t_1) - \int_{t_1}^{t}\left\{\rho(s)Q(s) - \left[\frac{\rho'(s)}{\alpha+1}\right]^{\alpha+1} \frac{r(s)}{[\rho(s)\sigma'(s)]^\alpha}\right\}\mathrm{d}s$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} w(t) = -\infty$, which contradicts $w(t) > 0$, then eq. (2) has not a positive solution. Similarly, then eq. (2) has not a negative solution. Therefore, eq. (2) is oscillatory.

Corollary 1. Assume that:

$$\beta \ge \alpha > 0, \quad \int_{t_0}^{\infty} \rho(s)Q(s)d\mathrm{d}s = \infty, \quad \int_{t_0}^{\infty} \frac{[\rho'(\mathrm{s})]^{\alpha+1}r(s)}{[\rho(s)\sigma'(s)]^\alpha}\mathrm{d}s = \infty$$

then eq. (2) is oscillatory.

Corollary 2. Assume that $\beta \geq \alpha > 0$, and the following conditions are hold:

$$(1)\ \lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{r(s)}\int_{s}^{\infty}Q(u)\mathrm{d}u\right]\frac{1}{\alpha}\mathrm{d}s=\infty$$

$$(2)\int_{t_0}^{\infty}\frac{[\rho'(\mathrm{s})]^{\alpha+1}r(s)}{\left[\rho(s)\sigma'(s)\right]^{\alpha}}\mathrm{d}s=\infty$$

$$(3)\ \underline{\lim}_{t\to\infty}\frac{[\rho(t)]^{\alpha+1}[\sigma'(t)]^{\alpha}}{[\rho'(t)]^{\alpha+1}r(t)}Q(t)>\left(\frac{1}{\alpha+1}\right)^{\alpha+1}$$

then eq. (2) is oscillatory.

Proof. From conditions (3), we have:

$$\frac{[\rho(t)]^{\alpha+1}[\sigma'(t)]^{\alpha}}{[\rho'(t)]^{\alpha+1}r(t)}Q(t)>\left(\frac{1}{\alpha+1}\right)^{\alpha+1}+\varepsilon,\quad \forall\varepsilon>0$$

$$\rho(t)Q(t)-\left(\frac{1}{\alpha+1}\right)^{\alpha+1}\frac{[\rho'(t)]^{\alpha+1}r(t)}{[\rho(t)\sigma'(t)]^{\alpha}}>\varepsilon\frac{[\rho'(t)]^{\alpha+1}r(t)}{[\rho(t)\sigma'(t)]^{\alpha}}$$

$$\int_{t_0}^{t}\left\{\rho(s)Q(s)-\left(\frac{1}{\alpha+1}\right)^{\alpha+1}\frac{[\rho'(s)]^{\alpha+1}r(s)}{[\rho(s)\sigma'(s)]^{\alpha}}\right\}\mathrm{d}s>\varepsilon\int_{t_0}^{t}\left\{\frac{[\rho'(s)]^{\alpha+1}r(s)}{[\rho(s)\sigma'(s)]^{\alpha}}\right\}\mathrm{d}s,\quad t>t_0$$

Letting $t\to\infty$, we get eq. (2) is oscillatory.

Theorem 3. Assume that $(H_1) \sim (H_4)$ and and $\alpha \geq \beta > 0$ hold, if for each $t \geq t_0$, there exist a function $\rho(t) \in C^1[I, (0, \infty)]$, $\rho'(t) \geq 0$ such that:

$$\lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{r(s)}\int_{s}^{\infty}Q(u)\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s=\infty,\quad \overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left\{\rho(s)Q(s)-\left[\frac{\rho'(s)}{\beta+1}\right]^{\beta+1}\frac{r(s)}{[\rho(s)\sigma'(s)]^{\beta}}\frac{1}{b^{\frac{\alpha-\beta}{\beta}}}\right\}\mathrm{d}s=\infty$$

where $Q(t) = \mathrm{q}(t)\{1 - p[\sigma(t)]\}^{\beta}$, $0 < b \leq z'(t)$, then eq. (2) is oscillatory.

Proof. Suppose that eq. (2) has a positive solution $x(t)$, Similarly (9), we have:

$$w'(t)\leq-\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\beta b^{\frac{\alpha-\beta}{\beta}}\sigma'(t)}{[\rho(t)r(t)]^{\frac{1}{\beta}}}[w(t)]^{\frac{\beta+1}{\beta}}$$

From Lemma, we have:

$$w'(t)\leq-\rho(t)Q(t)+\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left[\frac{\rho'(t)}{\rho(t)}\right]^{\beta+1}\frac{\rho(t)r(t)}{\beta^{\beta}[\sigma'(t)]^{\beta}b^{\frac{\alpha-\beta}{\beta}}}$$

$$w(t) \le w(t_1) - \int_{t_1}^{t} \left\{ \rho(s)Q(s) - \left[\frac{\rho'(s)}{\beta+1}\right]^{\beta+1} \frac{r(s)}{[\rho(s)\sigma'(s)]^{\beta} b^{\frac{\alpha-\beta}{\beta}}} \right\} \mathrm{d}s$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} w(t) = -\infty$, which contradicts $w(t) > 0$, then eq. (2) has not a positive solution. Similarly, then eq. (2) has not a negative solution. Therefore, eq. (2) is oscillatory.

Corollary 3. Assume that:

$$\alpha \ge \beta > 0, \quad \int_{t_0}^{\infty} \rho(s)Q(s)d\mathrm{d}s = \infty, \quad \int_{t_0}^{\infty} \frac{[\rho'(\mathrm{s})]^{\beta+1}}{[\rho(s)\sigma'(s)]^{\beta} b^{\frac{\alpha-\beta}{\beta}}} \mathrm{d}s = \infty$$

then eq. (2) is oscillatory.

Corollary 4. Assume that $\alpha \ge \beta > 0$, and the following conditions are hold:

$$(1)\ \lim_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{r(s)} \int_{s}^{\infty} Q(u)\mathrm{d}u \right]^{\frac{1}{\beta}} \mathrm{d}s = \infty$$

$$(2)\ \int_{t_0}^{\infty} \frac{[\rho'(\mathrm{s})]^{\beta+1} r(s)}{[\rho(s)\sigma'(s)]^{\beta} b^{\frac{\alpha-\beta}{\alpha}}} \mathrm{d}s = \infty$$

$$(3)\ \underline{\lim}_{t\to\infty} \frac{[\rho(t)]^{\beta+1}[\sigma'(t)]^{\beta}}{[\rho'(t)]^{\beta+1} r(t) b^{\frac{\alpha-\beta}{\beta}}} Q(t) > \left(\frac{1}{\beta+1}\right)^{\beta+1}$$

then eq. (2) is oscillatory.

Proof. From conditions (3), we have:

$$\frac{[\rho(t)]^{\beta+1}[\sigma'(t)]^{\beta}}{[\rho'(t)]^{\beta+1} r(t) b^{\frac{\alpha-\beta}{\beta}}} Q(t) > \left(\frac{1}{\beta+1}\right)^{\beta+1} + \varepsilon, \quad \forall \varepsilon > 0$$

that is:

$$\rho(t)Q(t) - \left(\frac{1}{\beta+1}\right)^{\beta+1} \frac{[\rho'(t)]^{\beta+1} r(t) b^{\frac{\alpha-\beta}{\beta}}}{[\rho(t)\sigma'(t)]^{\beta}} > \varepsilon \frac{[\rho'(t)]^{\beta+1} r(t) b^{\frac{\alpha-\beta}{\beta}}}{[\rho(t)\sigma'(t)]^{\beta}}$$

$$\int_{t_0}^{t} \left\{ \rho(s)Q(s) - \left(\frac{1}{\beta+1}\right)^{\beta+1} \frac{[\rho'(s)]^{\beta+1} r(s) b^{\frac{\alpha-\beta}{\beta}}}{[\rho(s)\sigma'(s)]^{\beta}} \right\} \mathrm{d}s > \varepsilon \int_{t_0}^{t} \left\{ \frac{[\rho'(s)]^{\beta+1} r(s) b^{\frac{\alpha-\beta}{\beta}}}{[\rho(s)\sigma'(s)]^{\beta}} \right\} \mathrm{d}s, \quad t > t_0$$

Letting $t \to \infty$, we get eq. (2) is oscillatory.

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