TICKHONOV BASED WELL-CONDITION ASYMPTOTIC WAVEFORM EVALUATION FOR DUAL-PHASE-LAG HEAT CONDUCTION

by

Sohel RANA, Jeevan KANESAN^{*}, Ahmed Wasif REZA, and Harikrishnan RAMIAH

Department of Electrical Engineering, Faculty of Engineering, University Malaya, Kuala Lumpur, Malaysia

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The Tickhonov based well-condition asymptotic waveform evaluation is presented here to study the non-Fourier heat conduction problems with various boundary conditions. In this paper, a novel Tickhonov based well-condition asymptotic waveform evaluation method is proposed to overwhelm ill-conditioning of the asymptotic waveform evaluation technique for thermal analysis and also presented for time-reliant problems. The Tickhonov based well-condition asymptotic waveform evaluation method is capable to evade the instability of asymptotic waveform evaluation and also efficaciously approximates the initial high frequency and delay similar as well-established numerical method, such as Runge-Kutta. Furthermore, Tickhonov based well-condition asymptotic waveform evaluation method is found 1.2 times faster than the asymptotic waveform evaluation and also 4 times faster than the traditional Runge-Kutta method.

Key words: Tickhonov well-condition asymptotic waveform evaluation, finite element method, heat conduction, boundary condition

Introduction

In earlier decades, the thermal analyses have been done by using traditional iterative manners. These iterative manners were undeniably very precise, but these methods are computationally expensive. Model based parameter estimation (MBPE) was made notorious to abate the computational convolution [1]. In 1990, asymptotic waveform evaluation (AWE) model was presented [2], which is a superior case of MBPE technique, and they are competent to show that AWE method is at least 3.33 times faster than the conventional frequency domain numerical methods. Unfortunately, AWE moment matching is ill-conditioned, because AWE technique is incompetent to forecast the delay and initial high frequencies in an accurate manner [3]. Then Lanczos [4], process was developed, and in this process, the linear systems are transformed into Pade approximation deprived of forming ill-conditioned moments. But if the systems are non-linear, researcher either can elucidate the problem by using ill-conditioned AWE technique, or they can renovate non-linear system into linear system. If they linearize the problems, either higher order term deserted or they should be familiarized superfluous degree of freedoms. To evade all those complications, a technique was introduced, entitled well-conditioned asymptotic waveform evaluation (WCAWE) [5]. Using

^{*} Corresponding author; e-mail: jievan@um.edu.my

WCAWE, moment matching process does not disregard higher order term and also avoids extra degree of sovereignty for non-linear systems. In WCAWE method, they acquaint with two correction terms to confiscate ill-condition of AWE moment matching, and this method was well-recognized to solve the frequency domain finite element exclusively for electromagnetic problems, where the simulation was carried out for different types of antennas. In WCAWE technique, [Z] matrix was picked randomly to find out the correction terms. In recent years, WCAWE method is used to solve the most challenging Helmholtz finite element model (FEM) [6].

The objectives of the present work are to deliberate about Fourier and non-Fourier heat conduction problems for diverse boundaries conditions. Various methods can be found to solve Fourier and non-Fourier heat conduction equation with different initial and boundary conditions [7-12]. They used various conventional iterative techniques, which are computationally expensive. In AWE method, the non-Fourier heat conduction models are needed to be converted into a linear equation [13, 14]. In this method, higher order term is deserted during the transformation of linear equation. Hence, the technique is not capable to forecast the tangible temperature responses. Recently, Tickhonov based well-condition asymptotic waveform evaluation technique (TWCAWE) was proposed for fast transient thermal analysis of non-Fourier heat conduction. The technique successfully approximates the temperature responses for single boundary condition, as reported in [15]. However, the method can not predict the temperature responses in case of different boundary conditions because the algorithm might breakdown in any specific situation. Therefore, in the current work, we propose TWCAWE method to investigate the Fourier and non-Fourier heat conduction in different boundary conditions, which implanted with Tickhonov regularization technique to enrich the immovability [16, 17]. In this proposed TWCAWE model, no need to renovate non-Fourier heat conduction equation into linear equation. In the current study, we are capable to find out the [Z] matrix mathematically instead of choosing randomly, which assists to find out the correction term efficiently. The results attained from TWCAWE method precisely matched with Runge-Kutta (R-K) results and also competent to remove all instabilities of AWE. Additionally, the method proposed in the current work is 1.2 times faster compared to AWE.

Mathematical formulation

The fundamental perception of the Fourier heat conduction model is infinite thermal wave propagation speed. The classical Fourier heat conduction law relates the heat flux vector, q, to the temperature gradient, $\nabla \theta$:

$$q = -K_{\rm c} \nabla \theta \tag{1}$$

where K_c is the thermal conductivity.

Hence, the previous traditional parabolic heat equation is symbolized by eq. (2) that originated from eq. (1):

$$\sigma \nabla \theta = \frac{\partial \theta}{\partial t} \tag{2}$$

where $\sigma = K_c/\rho c$, ρ and c are the thermal diffusivity, the mass density, and the specific heat capacity, respectively.

The traditional heat conduction equation, symbolized by eq. (1), is unable to explain some distinct cases of heat conduction, *e. g.*, near the absolute zero temperature and extreme

thermal gradient. Tzou *et al.* [3] suggested a non-Fourier heat transfer model that contains two phase delay denoted by eq. (3). Due to the presence of phase delay, this model is able to describe these distinct situations and also able to eradicate the dimness of the classical heat conduction model:

$$Q(r, t + \kappa_a) = -K_c \nabla \theta(r, t + \kappa_b)$$
(3)

where κ_b is the phase delay of longitudinal temperature gradient in regards to local temperature and κ_a is the phase delay of heat flux in regards to local temperature.

The model denoted by eq. (3) promises both phase delay κ_a and κ_b for fast heat transfer. The model is represented in a standardized 2-D hyperbolic equation specified by eq. (4):

$$\frac{\partial^2 \phi}{\partial \varepsilon^2} + \frac{\partial^2 \phi}{\partial \psi^2} + Z_b \frac{\partial^3 \phi}{\partial \beta \partial \varepsilon^2} + Z_b \frac{\partial^3 \phi}{\partial \beta \partial \psi^2} = \frac{\partial \phi}{\partial \beta} + Z_a \frac{\partial^2 \phi}{\partial \beta^2}$$
(4)

where

$$\phi = \frac{\theta - \theta_0}{\theta_w - \theta_0}, \qquad \beta = \frac{t}{\frac{l^2}{\sigma}}, \qquad \varepsilon = \frac{x}{l}, \qquad \psi = \frac{y}{h}, \qquad Z_a = \frac{\kappa_a}{\frac{l^2}{\sigma}}, \qquad Z_b = \frac{\kappa_b}{\frac{l^2}{\sigma}}$$

where β is the standardized time, Z_b – the standardized phase delay for temperature gradient, and Z_a – the standardized phase delay for heat flux. The length and width are given by l and h, respectively, while σ is the thermal diffusivity.

The FEM is a sturdy technique as it is able to explain multi-dimensional and diverse types of problems [18-22]. The FEM meshing was carried out using the rectangular element with nodes j, k, m, and o. The FEM based on Galerkin's weighted residual technique is applied on eq. (4) to achieve eq. (5) [3]:

$$\int_{\Delta} \left[N \right]^{T} \left[\frac{\partial^{2} \phi}{\partial \varepsilon^{2}} + \frac{\partial^{2} \phi}{\partial \psi^{2}} + Z_{b} \frac{\partial^{3} \phi}{\partial \beta \partial \varepsilon^{2}} + Z_{b} \frac{\partial^{3} \phi}{\partial \beta \partial \psi^{2}} - \frac{\partial \phi}{\partial \beta} - Z_{a} \frac{\partial^{2} \phi}{\partial \beta^{2}} \right] \partial \varepsilon \partial \psi = 0$$
(5)

Elemental matrices can be instigated for any type of model by using eq. (5).

The TWCAWE algorithm

Consider a model of physical phenomenon:

$$[\mathbf{K}](s)\vec{\mathbf{x}}(s) = \vec{\mathbf{F}}(s) \tag{6}$$

where [K](s) is the compound matrix, $\vec{F}(s)$ – the compound excitation vector, $\vec{x}(s)$ – the key vector, and $s = j2\pi f(j)$ is the imaginary notation, f – the frequency). Consider an extension point $s_0(s_0 = j2\pi f_0)$ and the Taylor series is [5]:

$$\sum_{n=0}^{b_1} (s - s_0)^n [\mathbf{K}]_n \vec{\mathbf{x}}(s) = \sum_{n=0}^{c_1} (s - s_0)^n \vec{\mathbf{F}}_n$$
(7)

where b_1 and c_1 are selected large enough so that no substantial higher order terms of $[K]_n$ and/or \vec{F}_n are trimmed.

The moments matching AWE subspaces for eq. (8) are rendering to eq. (7). The moments are:

$$m_{1} = [\mathbf{K}]_{0}^{-1} \vec{\mathbf{F}}_{0}$$

$$m_{2} = [\mathbf{K}]_{0}^{-1} (\vec{\mathbf{F}}_{1} - [\mathbf{K}]_{1} m_{1})$$

$$m_{3} = [\mathbf{K}]_{0}^{-1} (\vec{\mathbf{F}}_{2} - [\mathbf{K}]_{2} m_{2})$$

$$m_{q} = [\mathbf{K}]_{0}^{-1} \left(\vec{\mathbf{F}}_{q-1} - \sum_{d=1}^{q-1} [\mathbf{K}]_{d} m_{q-d} \right)$$
(8)

Table 1. Algorithm 1 (TWCAWE moments calculation)

$v_1 = [\mathbf{K}]_0^{-1} \vec{\mathbf{F}}_0$			
$Z_{[1,1]} = \ v_1\ $			
$w_1 = v Z_{[1,1]}^{-1}$			
for $r = 2, 3,, n$			
$v_{r} = [\mathbf{K}]_{0}^{-1} \left\{ \sum_{m=1}^{r-1} \left[\vec{\mathbf{F}}_{m} \vec{\mathbf{e}}_{1}^{T} J_{U_{1}}(r, m) - \left[-[\mathbf{K}]_{1} w_{r-1} - \sum_{m=2}^{r-1} [\mathbf{K}]_{m} w_{r-m} J_{U_{2}}(r, m) \vec{\mathbf{e}}_{r-m} \right] \right\}$			
for $p = 1, 2, 3,, (r-1)$ do			
$Z_{[p,r]} = w_p^H v_r$			
$v_n = v_n - Z_{[p,r]} w_p$			
$Z_{[r,r]} = v_r , \qquad w_r = v_r Z_{[r,r]}^{-1}$			

These moments are linearly dependent for trivial values of q as monotonous pre-multiplied by $[K]_0^{-1}$. To avoid these difficulties, moments are calculated in an alternative way in TWCAWE method, which is able to produce significant results for any value of q. Consider TWCAWE moment subspaces are $V = (v_1, v_2, ..., v_n)$. The TWCAWE moments (Algorithm 1) are calculated, as shown in tab. 1.

Here, we simplified certain scheme:

$$J_{U_w}(r, m) = \prod_{i=1}^m Z_{[i:r-m+i-1,i:r-m+i-1]}$$

and \vec{e}_r is the r^{th} unit vector, all its entries are 0 excluding the r^{th} , which is 1.

Breakdown of the TWCAWE

In the present work, TWCAWE method has been used to forecast the temperature response in diverse boundary conditions. In some specific circumstances, the algorithm might breakdown [6, 22, 23]. Assume the Taylor coefficient matrices are: $[K]_0$, $[K]_1 = [K]_0$, $[K]_2 \neq [K]_0$, and right hand side follows the identical pattern where the moments rendering to eq. (8) are:

$$m_{1} = [K]_{0}^{-1}\vec{F}_{0}$$

$$m_{2} = [K]_{0}^{-1}(\vec{F}_{1} - [K]_{1}m_{1})$$

$$m_{2} = [K]_{0}^{-1}(\vec{F}_{0} - [K]_{0}[K]_{0}^{-1}\vec{F}_{0}) = 0$$

$$m_{3} = [K]_{0}^{-1}(\vec{F}_{2} - [K]_{2}[K]_{0}^{-1}\vec{F}_{0}) \neq 0$$

The TWCAWE method, as represented in algorithm 1. For additional precision, the above cases produce the following:

(a)
$$v_1 = [K]_0^{-1} \vec{F}_0, \quad Z_{[1,1]} = ||v_1|| \Longrightarrow w_1 = v_1 Z_{[1,1]}^{-1}$$

(b)
$$v_2 = [K]_0^{-1} (F_0 \vec{e}_1^T J_{U_1}(2, 1) \vec{e}_1 - [K]_1 w_1) = \zeta$$

(c)
$$Z_{[1,2]} = w_1^H v_2 = \overline{\zeta} \Longrightarrow v_2 = v_2 - Z_{[1,2]} w_2 = \overline{\zeta}$$

(d)
$$Z_{[2,2]} = ||v_2|| = \vec{\zeta} \Rightarrow w_2 = v_2 Z_{[2,2]}^{-1} = \frac{\vec{\zeta}}{\vec{\zeta}}$$

where $\vec{\zeta}$ is a vector with entries below machine precision. To evade the breakdown, the modified TWCAWE moments calculations (algorithm 2) are shown in tab. 2.

$v_1 = [\mathbf{K}]_0^{-1} \vec{\mathbf{F}}_0$		
$Z_{[1,1]} = \ v_1\ $		
$w_1 = v Z_{[1,1]}^{-1}$		
for $r = 2, 3,, n$		
$k = 1, v_1 = w_1$		
$v_{r} = [\mathbf{K}]_{0}^{-1} \left\{ \sum_{m=1}^{r-1} \begin{bmatrix} \vec{\mathbf{F}}_{m} \vec{\mathbf{e}}_{1}^{T} J_{U_{1}}(r, m) - \\ -[\mathbf{K}]_{1} w_{r-1} - \sum_{m=2}^{r-1} [\mathbf{K}]_{m} w_{r-m} J_{U_{2}}(r, m) \vec{\mathbf{e}}_{r-m} \end{bmatrix} \right\}$		
for $p = 1, 2, 3,, (r - 1)$		
$Z_{[p,r]} = w_p^H v_r$		
$v_n = v_n - Z_{[p,r]} w_p$		
$Z_{[r,r]} = \ v_r\ , w_r = v_r Z_{[r,r]}^{-1}$		

Table 2. Algorithm 2 (modified TWCAWE moments calculation with no breakdown)

Tickhonov regularization scheme

The TWCAWE algorithm implements the Tickhonov regularization technique [16, 17] to amend the stiffness matrix marginally in mandate to condense the volatility problem. Contemplate a well-condition approximation problem, $Kx \approx y$, the residual $||Kx - y||_2$ appears to be the minimum depending on the optimal of $x = (K \cdot K)^{-1} Ky$. The singularity complaint of $(K \cdot K)^{-1}$ can be condensed by adding a term, which is indicated in eq. (9):

$$x = (K \cdot K + h_c^2[\mathbf{I}])^{-1} K y$$
(9)

where h_c is the regulation parameter which depends on the order of the equation, whereas [I] is the identical matrix. The family of this estimated inverse is defined by $C_b = (K \cdot K + h_c^2 [I])^{(-1)} K$. In TWCAWE, during moment calculation, inverse of $[K_0]$ matrix is supernumerary by C_h from WCAWE algorithm. Particulars of h_c and [I] can be found at [16].

Transient response

The nodal moments [a] can be taken out from the global moment matrix for any random node *i*:

$$[a_n]_i = [V]_i \tag{10}$$

The transient response for any random node *i* can be approximated by using Pade approximation, then supplementary streamlined to partial fractions [3], as exposed in eqs. (11)-(13):

$$T_i(s) = a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n$$
 (11)

$$=\frac{d_0 + d_1 s + \ldots + d_{s-1} s^{s-1}}{1 + c_1 s + \ldots + c_s s^s}$$
(12)

$$=\frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_q}{s-p_q}$$
(13)

Poles and residues can be originated by resolving eqs. (14)-(16):

$$\begin{bmatrix} a_{0} & a_{1} & a_{2} \dots & a_{q-1} \\ a_{1} & a_{2} & a_{3} \dots & a_{q} \\ \dots & \dots & \dots & \dots \\ a_{q-1} \dots & \dots & a_{2q-2} \end{bmatrix} \begin{bmatrix} c_{q} \\ c_{q-1} \\ \vdots \\ c_{1} \end{bmatrix} = - \begin{bmatrix} a_{q} \\ a_{q-1} \\ \vdots \\ a_{1} \end{bmatrix}$$
(14)

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$$\sum_{i=1}^{r} c_i p^i + 1 = 0 \tag{15}$$

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & p_q^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & p_q^{-2} \\ \vdots & \vdots & \vdots & \vdots \\ p_1^{-s} & p_2^{-s} & p_3^{-s} & \vdots & p_q^{-q} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ r_q \end{bmatrix} = - \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{q-1} \end{bmatrix}$$
(16)

The final transient response for any random node, *i* is:

$$T_{i}(t) = \sum_{i=1}^{q} \frac{r_{i}^{*}}{p_{i}^{*}} \left[\left(\vec{e}^{\operatorname{real}(p_{i}^{*})} \left\{ \cos[\operatorname{imag}(p_{i}^{*})]t + \sin[\operatorname{imag}(p_{i}^{*})]t \right\} \right) - 1 \right]$$
(17)

Boundary conditions

In the present work, the Fourier and non-Fourier heat transfer is investigated by considering 2-D rectangular models. The heat conduction mod-

el with dual phase delay has been explained here by using finite element analysis.

For heating flux, three distinctive temporal profiles, such as (1) instantaneous heat impulse, (2) constant heat imposed, and (3) periodic type are preferred because those boundary conditions are widely used in engineering problems. Figure 1 shows 2-D rectangular slabs meshed to generate the rectangular element. Diverse boundary conditions shown in fig. 2 are executed at the left edge of the slab.

Figure 1. The 2-D rectangular

Figure 1. The 2-D rectangular slab meshed to generate rectangular elements

Results and discussions

Observation of Fourier heat conduction

Figures 3 and 4 exemplify the Fourier temperature responses respecting the normalized time and normalized distance, correspondingly with instant heat imposed at the left edge of the boundary. The results demonstrate that, the thermal influence is sensed instantaneously throughout the system, if the surface of a material is heated. This leads to the simultaneous development of heat flux and temperature gradient. The classical Fourier law assumes instantaneous thermal equilibrium between the electrons and photons. Figures 3 and 4 also spectacles that, when instantaneous heat is executed at the left edge of the boundary, the same thermal effect is felt at the right edge of the slab instantly.



Figure 2. Boundary conditions; (a) instantaneous heat impulse, (b) periodic type, (c) constant heat imposed



Figure 3. Normalized temperature responses for Fourier heat conduction



Figure 4. Fourier temperature responses along the centre of the slab for different time

Observation of non-Fourier heat conduction

In the present work, three distinct boundary conditions are investigated for non-Fourier heat conduction. The phase delays are varied to spectacle the comparative significance of the wave term and the phonon-electron collaboration. In this work, to authenticate TWCAWE results, we compared the results with R-K outcomes. Figures 5-10 exemplify the temperature responses for non-Fourier heat conduction by using three typical temperature profiles. In these figures, we have compared the results of three different numerical techniques, such as R-K, AWE, and TWCAWE to prove the accuracy. In these comparisons, we have used similar experimental settings of total length and width of the slab, total number of nodes, thermal conductivity and imposed temperature for the mentioned methods. In this study, the fourth order the R-K method has been taken as an exact analytical technique as a benchmark, since the results obtained from this technique are accurate [3], but computationally expensive. Figures 5(a) and (b) display the temperature spreading of sudden heat executed on the left edge of the slab, also signifies the comparison of TWCAWE, R-K, and AWE methods for diverse values of Z_b at node 5 and node 59, respectively. It can be observed that, at node 5, initial fluctuation has been reduced, which helps to approximate the temperature behavior of other node accurately (e. g., node 59), as shown in fig. 5.



Figure 5. Normalized temperature responses along the centre of the slab for $Z_b = 0.5$, 0.05, and 0.0001 with instantaneous heat imposed; (a) at node 5, (b) at node 59

This evaluation spectacle that TWCAWE results absolutely match with the R-K outcome for $Z_b = 0.5$ and 0.05, but AWE results show inconsistency. For the instance of $Z_b = 0.0001$, TWCAWE congregates to similar steady-state as R-K. Before uniting to the steady-state, R-K results display convergence with TWCAWE, but AWE is incompetent to forecast the accurate temperature response as TWCAWE and R-K due to instability of AWE. It can be also observed from fig. 5 that, for TWCAWE method, the temperature behaviors are converged after normalized time of 0.25 and 0.31 in case of $Z_b = 0.05$ and $Z_b = 0.0001$, respectively, at node 5. In case of node 59, the temperature behaviors are converged after normalized time of 0.5 and 0.9 for $Z_b = 0.05$ and $Z_b = 0.0001$, respectively, because the node 59 is far from the boundary where temperature imposed, as shown in fig. 1.

Figures 6(a)-(d) display the non-Fourier temperature circulation respecting distance along the centre of the slab at time 0.005, 0.05, 0.1, and 0.5, correspondingly for TWCAWE. This temperature distribution with instant heat executed at boundary conditions for $Z_b = 0.5$, 0.05, and 0.0001 respectively. Immediate heat pulse is imposed on the left edge of

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Figure 6. Normalized temperature distribution along the centre of the slab for $Z_b = 0.5$, 0.05, and 0.0001 with instantaneous heat imposed; (a) at time 0.005, (b) at time 0.05, (c) at time 0.1, and (d) at time 0.5

the slab and the heat is drifting towards the other edge of the slab. Figure 6 also shows the evaluation of TWCAWE, R-K, and AWE outcomes. The evaluation shows TWCAWE results precisely adjacent to R-K solution, but AWE shows inconsistency.

Figures 7 and 8 point to the non-Fourier temperature responses for intervallic heat imposed at the left edge of the slab. Figures 7(a) and (b) signify the temperature dissemination with veneration in time for $Z_b = 0.5$, 0.05, and 0.0001 at node 5 and node 59, correspondingly. Figures 9 and 10 spectacle the temperature responses with persistent heat enacted at the left edge of the rectangular slab.



Figure 7. Normalized temperature response along the centre of the slab for $Z_b = 0.5$, 0.05, and $Z_b = 0.0001$ with periodic heat executed; (a) at node 5, (b) at node 59

These figures also epitomize the evaluation of TWCAWE, R-K and AWE. These figures spectacles that TWCAWE results utterly matched with R-K outcomes on behalf of $Z_b = 0.5$ and 0.05, but in this instance AWE results show divergence. In the case of



Figure 8. Normalized temperature distribution along the centre of the slab for three values of Z_b with periodic heat levied; (a) at time 0.005, (b) at time 0.05, (c) at time 0.1, and (d) at time 0.5



Figure 9. Normalized temperature response along for three values of Z_b with constant heat imposed; (a) at node 5, (b) at node 59

 $Z_b = 0.0001$, TWCAWE converges to a similar steady-state as R-K. Before converging to the steady-state, AWE outcomes are inept to envisage the accurate temperature response as TWCAWE. By investigating three forms of boundary condition results, it is perceived that, the heat conveyance promptness is penetrating to the way in which the boundary conditions are quantified and TWCAWE technique efficaciously imprecise delays precisely.

Table 3 exemplifies the simulation time disbursed by several methods deliberated in this work. The results spectacle AWE method is 3.33 times and TWCAWE is 4 times faster than



Figure 10. Normalized temperature dissemination along the centre of the slab for three values of Z_b with constant heat levied; (a) at time 0.005, (b) at time 0.05, (c) at time 0.1, and (d) at time 0.5

R-K technique. Additionally, from the mentioned results, we can determine that TWCAWE method is also 1.2 times faster than AWE technique.

Table 3. Required time for different methods Ratio with Time [s]

respect to R-K

1

3.33

4

	R-K	16
	AWE	4.8
WE tech-	TWCAWE	4

Method

Conclusion

This paper recommends TWCA nique to illustrate the non-Fourier heat conduc-

tion in diverse boundary conditions. There is no need to linearize the matrix equation and also no need to announce additional degree of freedom. Lastly, during moment calculation, we are competent to evaluate [Z] matrix mathematically for this problem. To perform fair comparison, similar experimental settings have been used for all three methods. The presented numerical comparison demonstrates that TWCAWE method is precise and wellconditioned since this technique can approximate the delay and initial high frequencies precisely. The numerical specimens also indicate that, the results are very subtle in the way in which the boundary conditions are quantified. Additionally, it is found that TWCAWE is 1.2 times faster than AWE, but 4 times faster than R-K. Moreover, the results are considerably superior than AWE.

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