NON-SIMILAR SOLUTIONS FOR NATURAL CONVECTION FROM A MOVING VERTICAL PLATE WITH A CONVECTIVE THERMAL BOUNDARY CONDITION

by

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In a recent paper by Makinde the effect of thermal buoyancy along a moving vertical plate with internal heat generation was considered. The plate thermal boundary condition was a convective condition with a heat transfer coefficient proportional to $x^{-1/2}$. *The fluid thermal expansion coefficient was proportional to* x^{-1}, *and the internal heat generation was assumed to decay exponentially across the boundary layer and proportional to* x^{-1} *in order that the problem accepts a similarity solution. In the present work, the same problem without heat generation is considered, with constant heat transfer coefficient and constant thermal expansion coefficient which is more realistic and has much more practical applications. The present problem is non-similar and results are obtained with the direct numerical solution of the governing equations. The problem is governed by the Prandtl number, the non-dimensional distance along the plate and a convective Grashof number, which is introduced for the first time. It is found that the wall shear stress, the wall heat transfer, and the wall temperature all increase with increasing distance and the wall temperature tends to one. The influence of the convective Grashof number is to increase the wall shear stress and the wall heat transfer and to reduce the wall temperature.*

Key words: *moving plate, boundary layer flow, heat transfer, convective parameter*

Introduction

The paper by Makinde [1] concerns the flow along a vertical plate moving with constant velocity in a calm fluid. The plate thermal boundary condition was a convective condition with a heat transfer coefficient proportional to $x^{-1/2}$ and the fluid thermal expansion coefficient being proportional to x^{-1}. In addition, internal heat generation exists which changes both along and across the plate (exponentially across the boundary layer and proportional to x^{-1} in the vertical direction). All the assumptions have been made in order that the problem accepts a similarity solution.

However, the assumption of a heat transfer coefficient varying along the plate as a function of $x^{-1/2}$ is not realistic and very difficult to be obtained in practice. Concerning the variation of the fluid thermal expansion coefficient there is a serious problem. The author mentions in the paper that the governing equations are based on the Boussinesq approximation. However, the Boussinesq approximation is based on the assumption that the fluid ther-

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mal expansion coefficient is constant and equal to that of the ambient fluid (see pages 266-267 in Schlichting and Gersten [2], and page 183 in Bejan [3]). Taking into account the quotes, the results of Makinde [1] have only theoretical value. In the present work we present results based on constant heat transfer coefficient and constant fluid thermal expansion coefficient which are compatible with real fluids. Non-similar solution is obtained for this problem which depends on the distance along the plate.

Problem definition and solution procedure

Consider the flow along a vertical semi-infinite plate with u and v denoting, respectively, the velocity components in x- and y-directions, where x is the co-ordinate along the plate, and y is the co-ordinate perpendicular to x (fig. 1 in [1]). For a steady, 2-D flow, the boundary layer equations are:

– continuity equation
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

– momentum equation
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\mathrm{g}\beta(T-T_\infty) \tag{2}$$

– energy equation
$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

subject to following boundary conditions at the plate:

$$u=u_w,\quad v=0,\ -k\frac{\partial T}{\partial y}=h_\mathrm{f}(T_\mathrm{f}-T)\ \text{ on }\ y=0,\qquad u\to 0,\ T\to T_\infty\qquad \text{as } y\to\infty \tag{4}$$

where ν is the fluid kinematic viscosity, β – the fluid thermal expansion coefficient, α – the fluid thermal diffusivity, k – the fluid thermal conductivity, and T – the fluid temperature. It is assumed that the plate is heated by convection from a fluid with constant temperature T_f with a heat transfer coefficient h_f.

Following Merkin and Pop [4], the following dimensionless quantities have been introduced:

$$X=\frac{\nu h_\mathrm{f}^2}{u_w k^2}x \tag{5}$$

$$Y=\frac{h_\mathrm{f}}{k}y \tag{6}$$

$$U=\frac{u}{u_w} \tag{7}$$

$$V=\frac{kv}{\nu h_\mathrm{f}} \tag{8}$$

$$\mathrm{Pr}=\frac{\nu}{\alpha} \tag{9}$$

$$\Theta=\frac{T-T_\infty}{T_\mathrm{f}-T_\infty} \tag{10}$$

$$\eta = \frac{Y}{\sqrt{X}} \tag{11}$$

$$\mathrm{Gr_c} = \frac{\mathrm{g}\beta(T_\mathrm{f} - T_\infty)k^2}{u_w h_\mathrm{f}^2 \nu} \tag{12}$$

The quantity given by eq. (12) is a new non-dimensional parameter, which is introduced here and named as convective Grashof number. Using the previous quantities the eq. (1)-(3) take the following dimensionless form:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{13}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} + \mathrm{Gr_c}\Theta \tag{14}$$

$$U\frac{\partial \Theta}{\partial X} + V\frac{\partial \Theta}{\partial Y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \Theta}{\partial Y^2} \tag{15}$$

The corresponding boundary conditions are:

$$U(X,0) = 1, \qquad V(X,0) = 0, \qquad U(X,\infty) = 0 \tag{16}$$

$$\frac{\partial \Theta}{\partial Y} = -(1-\Theta) \ \text{ on } \ Y = 0, \qquad \Theta = 0 \ \text{ as } \ Y \to \infty \tag{17}$$

Equations (13)-(15) represent a 2-D parabolic problem. Such a flow has a predominant velocity in the streamwise co-ordinate which in our case is the direction along the plate. In this type of flow convection always dominates the diffusion in the streamwise direction. Furthermore, no reverse flow is acceptable in the predominant direction. The solution of this problem is obtained using a finite difference algorithm as described by Patankar [5]. In order to obtain a complete form of both the temperature and velocity profile at the same cross section we used a non-uniform lateral grid. The ΔY takes small values near the surface (dense grid points near the surface) and increases along Y. A total of 500 lateral grid cells were used. It is known that the boundary layer thickness changes along X. For that reason the calculation domain must always be at least equal to or wider than the boundary layer thickness. In each case we tried to have a calculation domain wider than the real boundary layer thickness. This has been done by trial and error. If the calculation domain was thin the velocity and temperature profiles were truncated. In this case we used another wider calculation domain in order to capture the entire velocity and temperature profiles. The parabolic (space marching) solution procedure is described analytically in the textbook of Patankar [5] which *remains to this day a model of simplicity and clarity and one of the most coherent explications of the finite volume technique ever written* [6]. The above mentioned solution procedure is implicit and unconditionally stable [7, p. 276], has been used extensively in the literature and has been included in fluid mechanics and heat transfer textbooks [8, p. 364], [7, p. 271], and [9, p. 124]. The method has been used successfully in a series of papers by the present author, [10-14].

Results and discussion

The problem is non-similar and it is governed by three non-dimensional parameters: Prandtl number, convective Grashof number, and the non-dimensional distance along the plate expressed by the quantity X, eq. (5).

The most important parameters for this problem are the non-dimensional wall temperature and the non-dimensional wall shear stress, as well as the non-dimensional wall heat transfer defined:

$$U'(0) = \left[\frac{\partial U}{\partial \eta}\right]_{\eta=0} = \sqrt{X}\left[\frac{\partial U}{\partial Y}\right]_{Y=0} \tag{18}$$

$$\Theta'(0) = \left[\frac{\partial \Theta}{\partial \eta}\right]_{\eta=0} = \sqrt{X}\left[\frac{\partial \Theta}{\partial Y}\right]_{Y=0} \tag{19}$$

Before applying the current solution procedure to the present problem it was applied to the case considered by Merkin and Pop [4] in order to check its accuracy. The problem considered by [4] concerns the flow along a motionless plate inside a free stream without buoyancy. The results are shown in tab. 1.

Table 1. Values of dimensionless wall temperature $\Theta(0)$ for the non-similar case in forced convection flow with constant heat transfer coefficient without buoyancy for Pr = 1 (validation test)

X	$\Theta(0)$ Merkin and Pop [4]	$\Theta(0)$ Present method
0.001	0.062	0.062
0.124	0.457	0.446
1.867	0.784	0.780
8.335	0.895	0.892
110.335	1.0	0.970

Table 1 is a validation test of our numerical solution procedure. The values of Merkin and Pop [4] have been extracted from their fig. 4 due to lack of tabulated data in their work. Taking into account this fact the agreement between the results of [4] and the results obtained by the present solution procedure is satisfactory. Next results are presented for the non-similar case for mixed convection.

Table 2. Values of wall shear stress $U'(0)$, wall heat transfer $\Theta'(0)$, and wall temperature $\Theta(0)$ for different parameter values for the non-similar case with constant heat transfer coefficient and constant thermal expansion coefficient for Pr = 0.72

	$Gr_c = 1$			$Gr_c = 10$			$Gr_c = 100$		
X	$U'(0)$	$-\Theta'(0)$	$\Theta(0)$	$U'(0)$	$-\Theta'(0)$	$\Theta(0)$	$U'(0)$	$-\Theta'(0)$	$\Theta(0)$
0.001	–0.4437	0.0294	0.0473	–0.4437	0.0294	0.0473	–0.4416	0.0294	0.0472
0.01	–0.4383	0.0856	0.1415	–0.4362	0.0857	0.1417	–0.3654	0.0859	0.1401
0.1	–0.4209	0.2023	0.3603	–0.2394	0.2052	0.3511	–1.0929	0.2176	0.3119
1	–0.0501	0.3476	0.6525	2.3026	0.4155	0.5847	14.7546	0.5227	0.4775
10	3.2992	0.5775	0.8176	21.3404	0.8437	0.7335	111.873	1.2037	0.6198
100	25.460	1.0388	0.8962	142.444	1.6724	0.8329	745.816	2.6249	0.7377
1000	157.84	1.9159	0.9395	873.803	3.3140	0.8953	4631.61	5.3523	0.8309
10000	947.00	3.7261	0.9627	5231.32	6.1639	0.9383	27625.4	10.098	0.8990

From tab. 2, the following conclusions can be drawn. For a fixed value of the distance X, an increase of the convective Grashof number causes a reduction of the non-dimensional wall temperature and an increase in the wall shear stress and wall heat transfer.

From figs. 1 and 2, it is evident that as the convective Grashof number increases, the velocity and temperature boundary layer thicknesses decrease. For a fixed value of the convective Grashof number an increase of the distance *X* causes an increase of the non-dimensional wall temperature which tends to 1.

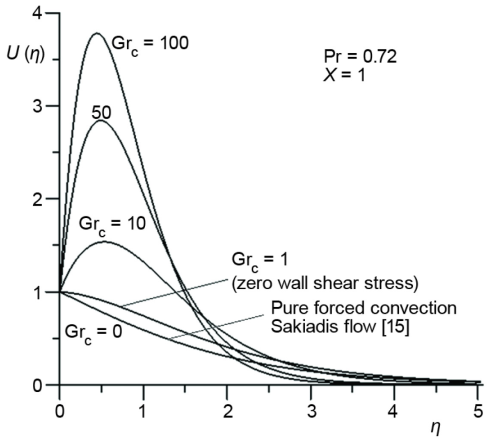

Figure 1. Velocity profiles for Pr = 0.72, *X* = 1, and different values of Gr$_c$

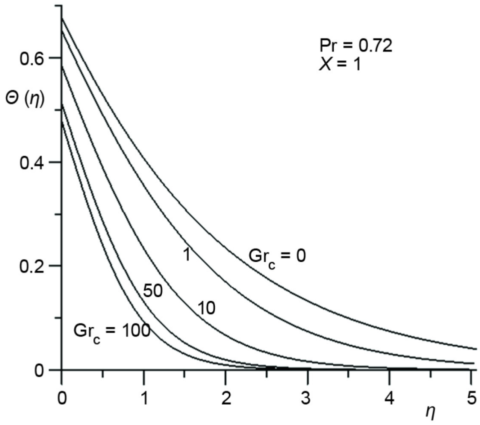

Figure 2. Temperature profiles for Pr = 0.72, *X* = 1, and different values of Gr$_c$

This means that the wall temperature tends to T_f, and the problem tends to become identical to the case of mixed convection along a vertical isothermal plate with plate temperature equal to T_f. In addition, as *X* increases, the wall shear stress and the wall heat transfer increase continuously. Figures 3 and 4 show velocity and temperature profiles at different distances *X* for Gr$_c$ = 10.

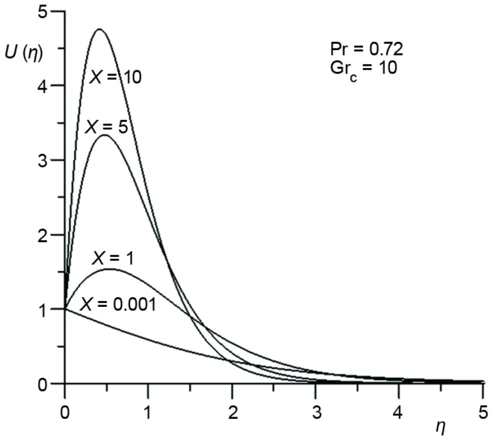

Figure 3. Velocity profiles for Pr = 0.72, Gr$_c$ = 10, and different values of *X*

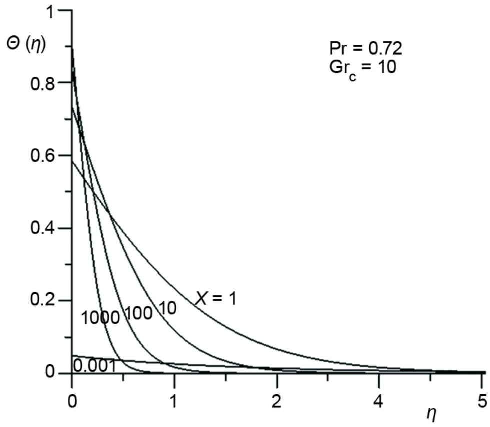

Figure 4. Temperature profiles for Pr = 0.72, Gr$_c$ = 10, and different values of *X*

It can be seen that as *X* increases the velocity and temperature increases and the profiles become thinner. At very low values of *X* the velocity is practically identical with that of the Sakiadis [15] flow (forced convection) and the corresponding wall shear stress is near the value –0.4437 which is the wall shear stress of the classical Sakiadis flow, (Schlichting and Gersten [2, page 177]. The influence of the Prandtl number is shown in figs. 5 and 6 where it is seen that, as the Prandtl number increases, the temperature decreases, the thermal boundary layer thickness decreases and the velocity decreases.

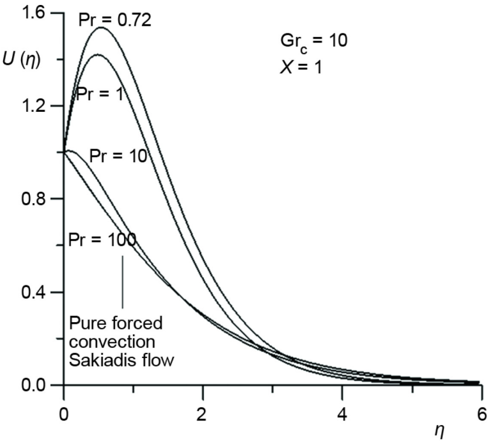

Figure 5. Velocity profiles for Gr_c = 10, X = 1, and different values of Prandtl number

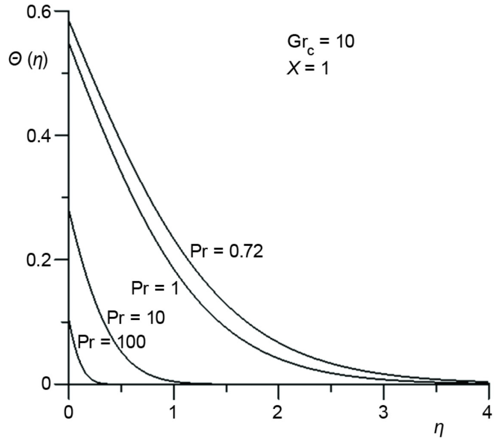

Figure 6. Temperature profiles for Gr_c = 10, X = 1, and different values of Prandtl number

Conclusions

The mixed convection problem along a vertical plate with convective boundary condition with constant heat transfer coefficient and constant thermal expansion coefficient has been investigated in this paper. The main conclusions can be summarized.

- When the convective Grashof number increases for specific value of X and Prandtl number the wall shear stress and the wall heat transfer increase, whereas the wall temperature and the velocity and thermal boundary layer thicknesses decrease. For Gr_c = 0 the problem changes from mixed convection to forced convection and the results of the momentum equation are identical with those of Sakiad flow [15].
- When the distance X is increased, the wall shear stress, the wall heat transfer and the wall temperature all increase, whereas the velocity and thermal boundary layer thicknesses decrease. The wall temperature tends to 1 as the distance X takes large values.
- An increase of the Prandtl number causes a reduction in the temperature and velocity which is accompanied by a thermal boundary layer reduction and an increase in the velocity boundary layer thickness. At very high Prandtl numbers (Pr → ∞) the buoyancy force near the plate tends to zero and the flow approaches asymptotically the pure forced convection state, Sakiadis flow [15].

Nomenclature

Gr_c – convective Grashof number, [–]
h_f – heat transfer coefficient, [$Wm^{-2}k^{-1}$]
k – thermal conductivity, [$Wm^{-1}K^{-1}$]
Pr – Prandtl number, [–]
T – fluid temperature, [K]
T_f – hot fluid temperature, [K]
U, V – dimensionless velocity components, [–]
u, v – velocity components, [ms^{-1}]
X, Y – dimensionless co-ordinates, [–]
x, y – Cartesian co-ordinates, [m]

Greek symbols

α – thermal diffusivity, [m^2s^{-1}]
β – thermal expansion coefficient, [K^{-1}]
η – similarity variable, [–]
Θ – dimensionless temperature, [–]
ν – kinematic viscosity, [m^2s^{-1}]

References

[1] Makinde, O. D., Similarity Solution for Natural Convection from a Moving Vertical Plate with Internal Heat Generation and a Convective Boundary Condition, *Thermal Science, 15* (2011), Suppl. 5, pp. S137-S143
[2] Schlichting, H., Gersten, K., *Boundary Layer Theory*, 9th ed., Springer, Berlin, 2003
[3] Bejan, A., *Convection Heat Transfer,* 3rd ed., John Wiley & Sons, N. J., USA, 2004
[4] Merkin, J., Pop, I., The Forced Convection Flow of an Uniform Stream over a Flat Surface with a Convective Surface Boundary Condition, *Commun Nonlinear Sci. Numer. Simulat., 16* (2011), 9, pp. 3602-3609
[5] Patankar, S. V., *Numerical Heat Transfer and Fluid Flow,* McGraw-Hill Book Company, New York, USA, 1980
[6] Acharya, S., Murthy, J., Foreword to the Special Issue on Computational Heat Transfer, *ASME Journal of Heat Transfer, 129* (2007), 4, pp. 405-406
[7] White, F., *Viscous Fluid Flow*, 3rd ed., McGraw-Hill, New York, USA, 2006
[8] Anderson, D., *et al., Computational Fluid Mechanics and Heat Transfer,* McGraw-Hill, New York, USA, 1984
[9] Oosthuizen, P., Naylor, D., *Introduction to Convective Heat Transfer Analysis,* McGraw-Hill, New York, USA, 1999
[10] Pantokratoras, A., Laminar Free Convection of Pure and Saline Water along a Heated Vertical Plate, *ASME Journal of Heat Transfer, 121* (1999), 3, pp. 719-722
[11] Pantokratoras, A., The Classical Plane Couette-Poiseuille Flow with Variable Fluid Properties, *ASME Journal of Fluids Engineering***,** *128* (2006), 5, pp. 1115-1151
[12] Pantokratoras, A., The Nonsimilar Laminar Wall Plume in a Constant Transverse Magnetic Field, *International Journal of Heat and Mass Transfer, 52* (2009a), 15, pp. 3873-3878
[13] Pantokratoras, A., The Nonsimilar Laminar Wall Jet with Uniform Blowing or Suction: New Results, *Mechanics Research Communications, 36* (2009b), 6, pp. 747-753
[14] Pantokratoras, A., Nonsimilar Aiding Mixed Convection along a Moving Cylinder in a Free Stream*, ZAMP, 61* (2010), 2, pp. 309-315
[15] Sakiadis, B. C., Boundary Layer Behavior on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface, *AIChE Journal, 7* (1961), 2, pp. 221-225

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