Open forum

ON THE SEMI-INVERSE METHOD AND VARIATIONAL PRINCIPLE

by

Xue-Wei LI*[a], *Ya LI*[a], *and Ji-Huan HE*[a,b]

[a] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China
[b] Nantong Textile Institute, Soochow University, Nantong, China

Short paper
DOI: 10.2298/TSCI1305565L

In this Open Forum, Liu et al. proved the equivalence between He-Lee 2009 variational principle and that by Tao and Chen (Tao, Z. L., Chen, G. H., Thermal Science, 17(2013), pp. 951-952) for one dimensional heat conduction. We confirm the correction of Liu et al.'s proof, and give a short remark on the history of the semi-inverse method for establishment of a generalized variational principle.

Key words: *variational principle, heat conduction, semi-inverse method, Lagrange multiplier, parameterized variational principle*

Introduction

Liu *et al*. proved that the following variational principle for 1-D heat conduction [1]:

$$\tilde{J}_{He\text{-}Lee} = \int_0^{t_0}\int_a^b \left[\alpha \left(\frac{\partial^2 T}{\partial x^2} + k^2 \frac{\partial T}{\partial t} + \lambda k^2 T \right) + \beta \right] \mathrm{d}x\mathrm{d}t \tag{1}$$

is equivalent to He-Lee 2009 variational principle [2]:

$$J_{He\text{-}Lee} = \int_0^{t_0}\int_a^b \left[\frac{\partial^2 T}{\partial x^2} + k^2 \frac{\partial T}{\partial t} + \lambda k^2 T \right] \mathrm{d}x\mathrm{d}t \tag{2}$$

and Tao-Chen 2013 variational principle [3]:

$$J_{Tao\text{-}Chen} = \int_0^{t_0}\int_a^b \left[\frac{\partial T}{\partial t} + \lambda T + \frac{1}{k^2}\frac{\partial^2 T}{\partial x^2} \right] \mathrm{d}x\mathrm{d}t \tag{3}$$

for all non-zero constants α and β. Liu *et al*.'s proof is straightforward and it is very easy for understanding. To show this, we consider a simple function:

$$y(x) = x^2 - x \tag{4}$$

Its extreme value is same with the following one:

* Corresponding author; e-mail: hejihuan@suda.edu.cn

$$y(x) = \alpha(x^2 - x) + \beta \tag{5}$$

for all non-zero constants α and β.

Tao and Chen applied the semi-inverse method proposed in 2007 [4], hereby we give a tutorial introduction to the method for beginners.

The semi-inverse method

To elucidate basic property of the semi-inverse method [4], we consider a 2-D incompressible and potential flow, its governing equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$\frac{\partial \Phi}{\partial x} = u, \quad \frac{\partial \Phi}{\partial y} = v \tag{7}$$

where u and v are velocities in x- and y-directions, respectively, Φ is the potential. There is a known variational principle for the problem, which is:

$$J(\Phi) = \iint \frac{1}{2}(u^2 + v^2)\mathrm{d}x\mathrm{d}y \tag{8}$$

which is subject to the constraints, eq. (7).

The general approach to establishment of a generalized variational principle is the Lagrange multiplier method [5]:

$$J(\Phi, u, v, \lambda_1, \lambda_2) = \iint \left[\frac{1}{2}(u^2 + v^2) + \lambda_1 \left(\frac{\partial \Phi}{\partial x} - u \right) + \lambda_2 \left(\frac{\partial \Phi}{\partial x} - u \right) \right] \mathrm{d}x\mathrm{d}y \tag{9}$$

where λ_1 and λ_2 are Lagrange multipliers.

Considering the fact that the Lagrange multipliers involved in eq. (9) are unknown, the semi-inverse method [4] is to replace the terms involving the Lagrange multipliers by an unknown function, F, in the form:

$$J(\Phi, u, v) = \iint \left[\frac{1}{2}(u^2 + v^2) + F \right] \mathrm{d}x\mathrm{d}y \tag{10}$$

where F is an unknown function of the variables u, v, Φ and/or their derivatives $F = F(u, v, \Phi, \Phi_x, \Phi_y, \ldots)$

We call eq. (10) a trial-functional. Making eq. (10) stationary with respect to u, v, and Φ:

$$\delta J(\Phi, u, v) = \iint \left[(u\delta u + v\delta v) + \frac{\delta F}{\delta u}\delta u + \frac{\delta F}{\delta v}\delta v + \frac{\delta F}{\delta \Phi}\delta \Phi \right] \mathrm{d}x\mathrm{d}y = \tag{11}$$

$$= \iint \left[\left(u + \frac{\delta F}{\delta u} \right)\delta u + \left(v + \frac{\delta F}{\delta v} \right)\delta v + \frac{\delta F}{\delta \Phi}\delta \Phi \right] \mathrm{d}x\mathrm{d}y = 0 \tag{11}$$

we have the following Euler-Lagrange equations:

$$u+\frac{\delta F}{\delta u}=0 \tag{12}$$

$$v+\frac{\delta F}{\delta v}=0 \tag{13}$$

$$\frac{\delta F}{\delta \Phi}=0 \tag{14}$$

where $\delta F/\delta \Phi$ is the variational derivative with respect to Φ, defined as:

$$\frac{\delta F}{\delta \Phi}=\frac{\partial F}{\partial \Phi}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \Phi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \Phi_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial \Phi_{xx}}\right)+\cdots \tag{15}$$

The above equations should be equivalent to eqs. (1) and (2). To this end, we set:

$$u+\frac{\delta F}{\delta u}=a\left(u-\frac{\partial \Phi}{\partial x}\right) \tag{16}$$

$$v+\frac{\delta F}{\delta v}=b\left(v-\frac{\partial \Phi}{\partial y}\right) \tag{17}$$

$$\frac{\delta F}{\delta \Phi}=c\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{18}$$

where *a, b*, and *c* are non-zero constants. From eqs. (16) and (17), we have:

$$F=\frac{1}{2}(a-1)u^2+\frac{1}{2}(b-1)v^2-au\frac{\partial \Phi}{\partial x}-bv\frac{\partial \Phi}{\partial y}+F_1 \tag{19}$$

where F_1 is an unknown function of Φ and/or its derivatives. Equations (18) and (19) imply that:

$$a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial y}+\frac{\delta F_1}{\delta \Phi}=c\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{20}$$

from which we can obtain the following relations:

$$F_1 = 0 \text{ and } a = b = k \tag{21}$$

Finally we obtain a parameterized variational principle, which reads:

$$J(\Phi,u,v)=\iint\left[\frac{1}{2}(u^2+v^2)+\frac{1}{2}(k-1)u^2+\frac{1}{2}(k-1)v^2-ku\frac{\partial \Phi}{\partial x}-kv\frac{\partial \Phi}{\partial y}\right]\mathrm{d}x\mathrm{d}y= \\ =-\iint k\left[\frac{1}{2}(u^2+v^2)+u\frac{\partial \Phi}{\partial x}-v\frac{\partial \Phi}{\partial y}\right]\mathrm{d}x\mathrm{d}y \tag{21}$$

where *k* is a non-zero constant.

Conclusions

Equation (1) is a parameterized variational principle, and we further confirm hereby that the variational principle by Tao and Chen is not a new one, but it is equivalent to He Lee 2009 variational principle. The development of the semi-inverse method was summarized in [6-9].

Acknowledgments

The work is supported by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), National Natural Science Foundation of China under grant No. 61303236 and No.11372205 and Project for Six Kinds of Top Talents in Jiangsu Province under grant No. ZBZZ-035, Science & Technology Pillar Program of Jiangsu Province under grant No. BE2013072.

References

[1] Fei, D.-D., *et al.*, A Short Remark on He-Lee's Variational Principle for Heat Conduction, *Thermal Science, 17* (2013), 5, pp. 1561-1563
[2] He, J.-H., Lee, E. W. M., A Constrained Variational Principle for Heat Conduction, *Physics Letters A, 373* (2009), 31, pp. 2614-2615
[3] Tao, Z. L., Chen, G. H., Remark on a Constrained Variational Principle for Heat Conduction, *Thermal Science, 17* (2013), 3, pp. 951-952
[4] He, J.-H., Semi-Inverse Method of Establishing Generalized Variational Principles for Fluid Mechanics with Emphasis on Turbomachinery Aerodynamics, *International Journal of Turbo & Jet-Engines, 14* (1997), 1, pp. 23-28
[5] He, J.-H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis,* 2012, 916793
[6] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B, 20* (2006), 10, pp. 1141-1199
[7] He, J.-H., An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering, *International Journal of Modern Physics B, 22* (2008), 21, pp. 3487-3578
[8] He, J.-H., Mo, L. F., Variational Approach to the Finned Tube Heat Exchanger Used in Hydride Hydrogen Storage System, *International Journal of Hydrogen Energy, 38* (2013), 36, pp. 16177-16178
[9] Qin, S. T., Ge, Y., A Novel Approach to Markowitz Portfolio Model without Using Lagrange Multipliers, *International Journal of Nonlinear Sciences and Numerical Simulation, 11* (2010), S, pp. 331-334

Paper submitted: November 26, 2013
Paper revised: November 30, 2013
Paper accepted: December 13, 2013