LIE SYMMETRY AND EXACT SOLUTION OF (2+1)-DIMENSIONAL GENERALIZED KADOMTSEV-PETVIASHVILI EQUATION WITH VARIABLE COEFFICIENTS

by

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The simple direct method is adopted to find Non-Auto-Backlund transformation for variable coefficient non-linear systems. The (2+1)-dimensional generalized Kadomtsev-Petviashvili equation with variable coefficients is used as an example to elucidate the solution procedure, and its symmetry transformation and exact solutions are obtained.

Key words: *(2+1)-dimensional Kadomtsev-Petviashvili equation, CK direct method, symmetry, variable coefficient*

Introduction

It is very important to obtain exact solutions of a given partial differential equation (PDE). The exact solutions, especially soliton solutions, may well describe various phenomena in our life. Based on the fact that soliton solutions are essentially of a localized nature, one can write the solitary wave solution of a non-linear equation as a polynomial of hyperbolic functions and transform it into a non-linear system of algebraic equations. Solitons are now studied in such diverse sciences as thermal science, fluid mechanics, biology, oceanography, meteorology, solid state physics, electronics, elementary particle physics, and cosmology [1]. Many kinds of methods have been proposed for obtaining explicit travelling solitary wave solutions, such as the inverse scattering method, the Hirota's method, the Backlund transformation, the Darboux transformation, and CK direct method.

The symmetry is an important tool in almost every branch of natural science. Lie's theory [2] is a standard method to find the Lie point symmetry group of a non-linear system. It had been used to find Lie point symmetry algebras and groups for almost all the known integrable systems successfully. Recently, it has been discovered that full finite Lie point symmetry transformation groups of constant coefficient integrable systems can be directly obtained with the help of gauge and space-time transformations of Lax pairs (or linear triads) without using the standard Lie algebra and Lie group theory [3]. For constant coefficient non-integrable systems, a direct method is also established to find Lie point subalgebras [4].

Lie symmetry and exact solution of (2+1)-dimensional generalized KP equation with variable coefficients

The Kadomtsev-Petviashvili (KP) equation:

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$$(u_t + 6uu_x + u_{xxx})_x + 3\kappa u_{yy} = 0 \tag{1}$$

with $\kappa = \pm 1$, is a two-dimensional generalization of the well-known Korteweg de-Vries (KdV) equation which can model several significant situations such as ones arising from the plasma physics [5] and surface water waves dynamics [6]. When $\kappa = -1$, the eq. 1 is usually called KPI, while for $\kappa = 1$, it is usually called KPII. The KP equation arises naturally in many other applications, such as gas dynamics and elsewhere. However, the physical situations in which the KP equation arises tend to be highly idealized, because of the assumption of constant coefficients, say, the propagation of small-amplitude surface waves in a fluid of constant depth. Currently, the variable-coefficient KP-typed (vcKP-typed) equations have attracted extensive attention for their realistic descriptions of wide range of physical applications containing the propagation of the two-dimensional dust-acoustic wave in the dusty plasma consisting of cold dust particles, an unmagnified, collisionless, isothermal electrons and two-temperature ions, surface waves through shallow seas and marines straits of varying width and depth with non-vanishing vorticity, and so on. The variable coefficient generalizations of the KP (GVCKP) equation provide a description of surface waves in a more realistic situation than the KP equation itself. The additional terms and the variable coefficients make it possible to treat straits of varying width and depth, variable density and to take vorticity into account. In recent years, the GVCKP equations have widely been studied from different perspectives such as symmetries, Backlund transformations, solitary solutions, soliton interactions, soliton-like solutions, Grammian form solutions, *etc*. [7-18]. The dimensionless form of the (2+1)-dimensional generalized KP equation with variable coefficients is given by:

$$[u_t + f(t)u_x u + g(t)u_{xxx}]_x + h(t)u_x + p(t)u_{yy} = 0 \tag{2}$$

where $f(t)$, $g(t)$ and $p(t)$ are non-zero functions. It is shown that the variable-coefficient functions for the GVCKP equation must satisfy the constraints to pass the Painleve tests for complete integrability. Exact single-soliton-like solutions of eq. 1 with $f(t) = 6$, $g(t) = 1$, have been investigated in [12-14].

At first, let:

$$u = \alpha + \beta U(\xi, \eta, \tau) \tag{3}$$

where $\alpha, \beta, \xi, \eta,$ and τ are the functions of $x, y,$ and $t.$

Substituting eq. (3) into eq. (2) and requiring $U(\xi, \eta, \tau)$ being also a solution of the (2+1)-dimensional KPII equation:

$$(U_\tau + 6UU_\xi + 6U_{\xi\xi\xi})_\xi + 3U_{\eta\eta} = 0 \tag{4}$$

but with the independent variables (eliminating $U_{\xi\eta}$ by means of the (2+1)-dimensional KPII equation), we have:

$$-4g(t)\beta\tau_x{}^3\xi_x U_{\xi 10} + F(x, y, t, U, U_\xi, U_\eta, \ldots) = 0 \tag{5}$$

where $U_{\xi\eta} = \partial_\xi^\eta U$, and F is the complexity function of U and independent of $U_{\xi 10}$. Equation 5 means:

$$\tau_x = 0 \tag{6}$$

Substituting eq. (6) into eq. (5):

$$p(t)\beta\tau_y{}^2 U_{\tau\tau} + F(x, y, t, U, U_\xi, U_\eta, \ldots) = 0 \tag{7}$$

which means:

$$\tau_y = 0 \tag{8}$$

Substituting eq. (8) into eq. (7) we can get $g(t)\beta\eta_x^{\,2}U_\eta^{\,4} + F(x, y, t, U, U_\xi, U_\eta, \ldots) = 0$, which means $\eta_x = 0$.

After tedious calculation, we have:

$$\tau = c^3\int g(t)\mathrm{d}t, \quad \beta = \frac{6g(t)c^2}{f(t)}, \quad \eta = \frac{\sqrt{3p(t)g(t)}c^2 y}{p(t)} + b(t) \tag{9}$$

$$\xi = cx + F(t) + \frac{3y^2c^2p_t(t)g(t) - 3y^2c^2p(t)g_g(t) - 4\sqrt{3}yb_t(t)p(t)\sqrt{p(t)g(t)}}{24cp(t)^2g(t)} \tag{10}$$

$$\alpha = \frac{9c^4\sqrt{p(t)g(t)}y^2\left[p_t(t)^2g(t)^2 - p(t)^2g_t(t)^2\right]}{48c^4p(t)^3g(t)^2\sqrt{p(t)g(t)}f(t)} - \frac{2c^2y\sqrt{3}b_t(t)g_t(t) + b_t(t)^2\sqrt{p(t)g(t)}}{12c^4g(t)\sqrt{p(t)g(t)}f(t)} +$$

$$+\frac{y^2\left[p(t)g_{tt}(t) - p_{tt}(t)g(t)\right]}{8p(t)^2g(g)f(t)} + \frac{y\sqrt{2}b_{tt}(t)}{6c^2\sqrt{p(t)g(t)}f(t)} - \frac{F_t(t)}{cf(t)} \tag{11}$$

with condition:

$$3p(t)^2g_t(t)^2 - 3g(t)^2p_t(t)^2 + 2p(t)g(t)^2p_{tt}(t) - 2p(t)^2g(t)g_{tt}(t) = 0 \tag{12}$$

where

$$p(t) = c_2\exp\int\left[\frac{4f_t(t)}{f(t)} - 4h(t) - \frac{3g_t(t)}{g(t)}\right]\mathrm{d}t$$

and $b(t)$, $F(t)$ are arbitrary functions of t, c, c_2 are arbitrary constants. Obviously we have the following theorem.

Theorem

If $U = U(\xi, \eta, \tau)$ is a solution of (2+1)-dimensional KPII eq. (1), then $u(x, y, t) = \alpha + \beta U(\xi, \eta, \tau)$, where τ, ξ, η, α, and β are decided by eqs. (9)-(12) is also the solution of eq. (2).

For simplicity, we use the solitary wave solution of eq. 1:

$$u(\xi, \eta, \tau) = \frac{1}{2}(k_2 - k_1)^2\operatorname{sech}^2\frac{\theta_2 - \theta_1}{2}$$

which gives the corresponding solution of eq. 2:

$$U(x, y, t) = \alpha + \beta\frac{1}{2}(k_2 - k_1)^2\operatorname{sech}^2\frac{\theta_2 - \theta_1}{2}$$

with eqs. (9)-(12).

Conclusions

In summary, we have utilized the simple direct method to find Non-Auto-Backlund Transformation between variable coefficient non-linear systems and constant coefficient ones.

Then taking advantages of the Non-Auto-Backlund Transformation and the known results of the constant coefficient non-linear equations, all kinds of the new solutions of the variable coefficient non-linear systems can be generated. The method can be readily extended to higher dimensional PDE.

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