

## HOMOTOPY PERTURBATION METHOD FOR VISCOUS HEATING IN PLANE COUETTE FLOW

by

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*In this paper, the problem of viscous heating in plane Couette flow is considered by the homotopy perturbation method. The non-linear terms are expanded to Taylor series of the homotopy parameter. The obtained solutions are shown graphically and are compared with the exact solutions. The obtained results illustrate the efficiency and convenience of the method.*

*Key words: homotopy perturbation method, homotopy equation, Taylor series, plane Couette flow, viscous dissipation*

### Introduction

We consider the steady flow of an incompressible Newtonian fluid between two infinite parallel plates, one of which is moving. The governing equations of the problem are [1]:

$$\frac{d}{dx} \left( \mu \frac{dv}{dx} \right) = 0, \quad \frac{d}{dx} \left( k \frac{dT}{dx} \right) + \mu \left( \frac{dv}{dx} \right)^2 = 0 \quad (1)$$

with the boundary conditions:

$$x = 0: \quad v = 0, T = T_0, \quad x = b: \quad v = V, T = T_0 \quad (2)$$

where  $v$  is the axial velocity,  $T$  – the temperature,  $\mu$  – the viscosity,  $k$  – the thermal conductivity,  $b$  – the distance of the two infinite parallel plates,  $T_0$  – the temperature around the two plates, and  $V$  – the velocity of the moving plate. Here, one consider constant thermal conductivity  $k = k_0$  and viscosity with exponential temperature dependence,  $\mu = \mu_0 \exp[-\alpha(T - T_0)]$ , where  $\alpha$  is a constant. Through the following dimensionless transformations:

$$\theta = \frac{T - T_0}{T_0}, \quad y = \frac{x}{b}, \quad u = \frac{v}{V}, \quad \beta = \alpha T_0, \quad \varepsilon = \frac{\mu_0 V^2}{k_0 T_0} \quad (3)$$

the governing eqs. (1) and the boundary conditions (2) are rewritten as:

$$\frac{d}{dy} \left( \exp(-\beta\theta) \frac{du}{dy} \right) = 0, \quad \frac{d^2\theta}{dy^2} + \varepsilon \exp(-\beta\theta) \left( \frac{du}{dy} \right)^2 = 0 \quad (4)$$

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with

$$y = 0: u = 0, \theta = 0, \quad y = 1: u = 1, \theta = 0 \quad (5)$$

In this paper, a powerful analysis technique, the homotopy perturbation method (HPM), is employed to solve the problem. The HPM was proposed by He [2, 3] and was further developed and improved by himself [4-9]. For the effectiveness and convenience of HPM, many other mathematicians and engineers are attracted to study its improvement, convergence and applications in many areas. A new modification of HPM was proposed by adding and subtracting a linear term in the considered original equations in [10]. Two algorithms for constructing the homotopy equation were given and applied to solve quadratic Riccati differential equation of fractional order in [11]. Combination of the Laplace transform and HPM was studied in [12]. Combination of the variational iteration method and HPM was studied in [13]. HPM was improved by truncating the infinite series corresponding to the first-order approximate solution before introducing this solution in the second-order linear differential equation in [14]. The multistage homotopy-perturbation method was proposed and used to solve the Lorenz system in [15]. A new extended homotopy perturbation method was proposed in [16]. The convergence of HPM was investigated in [17-19]. The applications of the HPM in many areas were also investigated such as the high-order boundary value problems [20-23], Static pull-in instability [24], Lane-Emden type singular IVPs problem [25], integro-differential equations [26], inverse problem [27], Hamilton-Jacobi-Bellman equation [28], heat transfer problem [29, 30] *et al.*

In [29], this problem was considered by the same technique. In order to improve those results of [29], we construct the homotopy equation only based on one of equations instead of all equations as [29]. Meanwhile, the non-linear terms are expanded to Taylor series of the homotopy parameter  $p$ . The obtained solutions are shown graphically and are compared with the exact solutions. The computational results show that the approximate solution obtained in this paper has higher accuracy than that in [29].

### Solution by HPM

Through denoting the non-linear terms as:

$$N_1[\theta, u] = \exp(-\beta\theta) \frac{du}{dy}, \quad N_2[\theta, u] = \exp(-\beta\theta) \left( \frac{du}{dy} \right)^2$$

the eqs. (4) are rewritten as:

$$\frac{d}{dy} N_1[\theta, u] = 0, \quad \frac{d^2\theta}{dy^2} + \varepsilon N_2[\theta, u] = 0 \quad (6)$$

Following the HPM, the expressions of  $u(y)$  and  $\theta(y)$  are assumed:

$$\theta(y) = \theta_0(y) + p\theta_1(y) + \dots + p^n\theta_n(y) + \dots, \quad u(y) = u_0(y) + pu_1(y) + \dots + p^nu_n(y) + \dots \quad (7)$$

The non-linear terms are expanded to Taylor series with respect to  $p$  on  $p = 0$ :

$$N_1[\theta, u] = \sum_{n=0}^{\infty} A'_n p^n, \quad N_2[\theta, u] = \sum_{n=0}^{\infty} A''_n p^n$$

where

$$A'_n = \frac{1}{n!} \frac{d^n}{dp^n} \left[ N_1 \left( \sum_{n=0}^{\infty} \theta_n p^n, \sum_{n=0}^{\infty} u_n p^n \right) \right]_{p=0}, n = 0, 1, 2, \dots, \quad (8)$$

$$A''_n = \frac{1}{n!} \frac{d^n}{dp^n} \left[ N_2 \left( \sum_{n=0}^{\infty} \theta_n p^n, \sum_{n=0}^{\infty} u_n p^n \right) \right]_{p=0}, n = 0, 1, 2, \dots.$$

Thus, the eqs. (6) are rewritten:

$$\frac{d}{dy} \sum_{n=0}^{\infty} A'_n p^n = 0 \quad (9)$$

$$\frac{d^2 \theta}{dy^2} + \varepsilon \sum_{n=0}^{\infty} A''_n p^n = 0 \quad (10)$$

The homotopy equation is constructed only based on the eq. (10):

$$\frac{d^2 \theta}{dy^2} + p \varepsilon \sum_{n=0}^{\infty} A''_n p^n = 0 \quad (11)$$

Substituting the expressions of solutions (7) to the eq. (9) and the homotopy eq. (11), then setting the coefficients of  $p^i$  ( $i = 0, 1, \dots, n, \dots$ ) are zero, we obtain a series of linear boundary problems:

$$p^0 : \begin{cases} \theta_0''(y) = 0 \\ \theta_0(0) = 0, \quad \theta_0(1) = 0 \end{cases}$$

$$p^0 : \begin{cases} u_0''(y) - \beta u_0'(y) \theta_0'(y) = 0 \\ u_0(0) = 0, \quad u_0(1) = 1 \end{cases}$$

$$p^1 : \begin{cases} \varepsilon e^{-\beta \theta_0(y)} [u_0'(y)]^2 + \theta_1''(y) = 0 \\ \theta_1(0) = 0, \quad \theta_1(1) = 0 \end{cases}$$

$$p^1 : \begin{cases} \beta^2 \theta_1(y) u_0'(y) \theta_0'(y) - \beta u_1'(y) \theta_0'(y) - \beta u_0'(y) \theta_1'(y) - \\ - \beta u_0''(y) \theta_1(y) + u_1''(y) = 0 \\ u_1(0) = 0, \quad u_1(1) = 0 \end{cases}$$

...

Solving these linear problems, the solutions are obtained:

$$\theta_0 = 0$$

$$\theta_1 = \frac{\varepsilon y}{2} - \frac{y^2 \varepsilon}{2}$$

$$\theta_2 = \frac{1}{24} y[y(y^2 - 2y + 2)\beta\epsilon^2 - \beta\epsilon^2]$$

$$u_0 = y$$

$$u_1 = -\frac{1}{6}y^3\beta\epsilon + \frac{1}{4}y^2\beta\epsilon - \frac{y\beta\epsilon}{12}$$

$$u_2 = -\frac{1}{960}(1-2y)^5\beta^2\epsilon^2 - \frac{1}{480}y\beta^2\epsilon^2 + \frac{\beta^2\epsilon^2}{960}$$

.....

Thus the  $n$ -order approximate solutions of  $\theta(y)$  and  $u(y)$  are obtained by let  $p = 1$ :

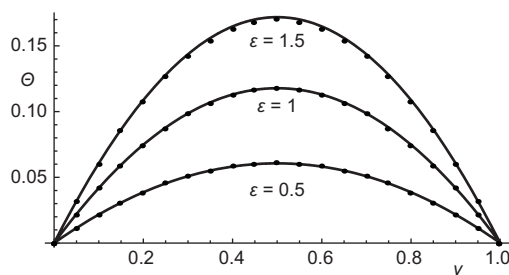
$$\theta(y) \approx \Xi_n(y) = \theta_0(y) + \theta_1(y) + \dots + \theta_{n-1}(y), \quad u(y) \approx U_n(y) = u_0(y) + u_1(y) + \dots + u_{n-1}(y)$$

These approximate solutions may be compared with the exact solution of [1] which is given by:

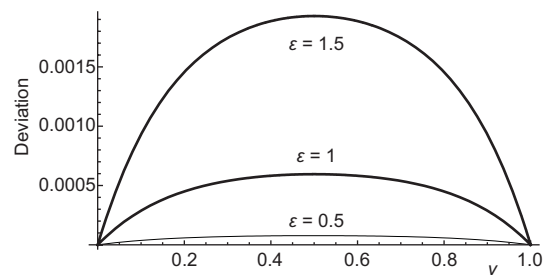
$$\exp[\beta\theta(y)] = \left(\frac{\epsilon\beta}{8} + 1\right) \cosh^{-2} \left[ (2y-1) \operatorname{arcsinh} \sqrt{\frac{\epsilon\beta}{8}} \right],$$

$$u = \frac{1}{2} \left\{ \sqrt{1 + \frac{8}{\epsilon\beta}} \tanh \left[ (2y-1) \operatorname{arcsinh} \sqrt{\frac{\epsilon\beta}{8}} \right] + 1 \right\}$$

This comparison of  $\theta$  is given in fig.1 when  $\beta = 1$  and a range of values of  $\epsilon$ . The graphs of the deviation of the present results from the exact solution are depicted in fig. 2 when  $\beta = 1$  and a range of values of  $\epsilon$ .



**Figure 1. Temperature distribution in a plane Couette flow,  $\beta = 1$ .**  
(Dots: the 3-order approximate solutions in the present work, Solid line: the exact solution)



**Figure 2. Deviation of the 3-order approximate solution from the exact solution,  $\beta = 1$**

## Conclusions

In this paper the homotopy equation is constructed based on one of two governing equations. The HPM is also combined with Taylor series. The approximate solutions of the problem of viscous heating in plane Couette flow are obtained with high accuracy. The efficiency and convenience of the HPM have confirmed again once.

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## References

- [1] Turian, R. M., Bird, R. B., Viscous Heating in the Cone-and-Plate Viscometer II, *Chem. Eng. Sci.*, 18 (1963), 11, pp. 689-696
- [2] He, J.-H., Homotopy Perturbation Technique, *Comput. Methods Appl. Mech. Eng.*, 178 (1999), 3-4, pp. 257-262
- [3] He, J.-H., A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non-Linear Problems, *Int. J. Nonlinear Mech.*, 35 (2000), 1, pp. 37- 43
- [4] He, J.-H., An Elementary Introduction to the Homotopy Perturbation Method, *Comput. Math. Appl.*, 57 (2009), 3, pp. 410-412
- [5] He, J.-H., The Homotopy Perturbation Method for Nonlinear Oscillators with Discontinuities, *Appl. Math. Comput.*, 151 (2004), 1, pp. 287-292
- [6] He, J.-H., Homotopy Perturbation Method for Solving Boundary Value Problems, *Phys. Lett. A*, 350 (2006), 1-2, pp. 87-88
- [7] He, J.-H., New Interpretation of Homotopy Perturbation Method, *Int. J. Mod. Phys. B*, 20 (2006), 18, pp. 2561-2568
- [8] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B*, 20 (2006), 10, pp. 1141-1199
- [9] He, J.-H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstr. Appl. Anal.*, 2012 (2012), pp. 1-130
- [10] Ganji, D. D., *et al.*, A New Modification of He's Homotopy Perturbation Method for Rapid Convergence of Nonlinear Undamped Oscillators, *J. Appl. Math. Comput.*, 30 (2009), 1-2, pp. 181-192
- [11] Odibat, Z., Momani, S., Modified Homotopy Perturbation Method: Application to Quadratic Riccati Differential Equation of Fractional Order, *Chaos Soliton Fract.*, 36 (2008), 1, pp. 167-174
- [12] Madania, M., *et al.*, On the Coupling of the Homotopy Perturbation Method and Laplace Transformation, *Math. Comput. Modelling*, 53 (2011), 9-10, pp. 1937-1945
- [13] Noor, M. A., Mohyud-Din, S. T., Variational Homotopy Perturbation Method for Solving Higher Dimensional Initial Boundary Value Problems, *Math. Probl. Eng.*, 2008 (2008), pp. 1-11
- [14] Belendez, A. *et al.*, Application of a Modified He's Homotopy Perturbation Method to Obtain Higher-order Approximations of an  $x^{1/3}$  Force Nonlinear Oscillator, *Phys. Lett. A*, 371 (2007), 5-6, pp. 421-426
- [15] Chowdhury, M. S. H., *et al.*, The Multistage Homotopy-Perturbation Method: a Powerful Scheme for Handling the Lorenz System, *Chaos Soliton Fract.*, 40 (2009), 4, pp. 1929-1937
- [16] Wang, F., *et al.*, A new Extended Homotopy Perturbation Method for Nonlinear Differential Equations, *Math. Comput. Modelling*, 55 (2012), 3-4, pp. 1471-1477
- [17] Biazar, J., Ghazvini, H., Convergence of the Homotopy Perturbation Method for Partial Differential Equations, *Nonlinear Anal. Real World Appl.*, 10 (2009), 5, pp. 2633-2640
- [18] Hetmaniok, E., *et al.*, Convergence and Error Estimation of Homotopy Perturbation Method for Fredholm and Volterra Integral Equations, *Appl. Math. Comput.*, 218 (2012), 1, pp. 10717-10725
- [19] Jafari, H., *et al.*, Convergence of Homotopy Perturbation Method for Solving Integral Equations, *Thai J. Math.*, 8 (2010), 3, pp. 511-520
- [20] Noor, M. A., Mohyud-Din, S. T., Homotopy Perturbation Method for Solving Sixth-Order Boundary Value Problems, *Comput. Math. Appl.*, 55 (2008), 12, pp. 2953-2972
- [21] Noor, M. A., Mohyud-Din, S. T., An Efficient Algorithm for Solving Fifth-Order Boundary Value Problems, *Math. Comput. Modelling*, 45 (2007), 7-8, pp. 954-964
- [22] Golbabai, A., Javidi, M., Application of Homotopy Perturbation Method for Solving Eighth-Order Boundary Value Problems, *Appl. Math. Comput.*, 191 (2007), 2, pp. 334-346
- [23] Mohyud-Din, S. T., Noor, M. A., Homotopy Perturbation Method for Solving Fourth-Order Boundary Value Problems, *Math. Probl. Eng.*, 2007 (2007), pp. 1-15
- [24] Mojahedi, M., *et al.*, Static Pull-in Analysis of Electrostatically Actuated Microbeams using Homotopy Perturbation Method, *Appl. Math. Modelling*, 34 (2010), 4, pp. 1032-1041

- [25] Yıldırım, A., Ozi, S. T., Solutions of Singular IVPs of Lane-Emden Type by Homotopy Perturbation Method, *Phys. Lett. A*, 369 (2007), 1-2, pp. 70-76
- [26] Yıldırım, A., Solution of BVPs for Fourth-Order Integro-Differential Equations by Using Homotopy Perturbation Method, *Comput. Math. Appl.*, 56 (2008), 12, pp. 3175-3180
- [27] Shakeri, F., Dehghan, M., Inverse Problem of Diffusion Equation by He's Homotopy Perturbation Method, *Phys. Scr.*, 75 (2007), 4, pp. 551-556
- [28] Saberi, Nik, *et al.*, An Approximate-Analytical Solution for the Hamilton-Jacobi-Bellman Equation via Homotopy Perturbation Method, *Appl. Math. Modelling*, 36 (2012), 11, pp. 5614-5623
- [29] Ganji, D. D., *et al.*, Application of Homotopy-Perturbation Method for Systems of Nonlinear Momentum and Heat Transfer Equations, *Heat. Transf. Res.*, 38 (2007), 4, pp. 361-379
- [30] Ganji, D. D., *et al.*, Determination of Temperature Distribution for Annular Fins with Temperature Dependent Thermal Conductivity by HPM, *Thermal Science*, 15 (2011), 1, pp. 111-115