A COMPARISON OF VARIOUS BASIS FUNCTIONS BASED ON MESHLESS LOCAL PETROV-GALERKIN METHOD FOR LINEAR STABILITY OF CIRCULAR JET

by

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Various basis functions based on Fourier-Chebyshev Petrov-Galerkin spectral method are described for computation of temporal linear stability of a circular jet. Basis functions presented here are exponentially mapped Chebyshev functions. There is a linear dependence between the components of the perturbation vector field, and there are only two degrees of freedom for the perturbation continuum equation. According to the principle of permutation and combination, the basis function has three basic forms, i. e., the radial, azimuthal or axial component, respectively. The results show that three eigenvalues for various cases are consistent, but there is a preferable basis function for numerical computation.

Key words: *hydrodynamic stability, circular jet, co-ordinates transformation, spectral-Galerkin method*

Introduction

Jets are important in many practical applications, *e. g.*, pulverized coal combustion and spray combustion [1-4], jet propulsion, atomization and spray [5], environmental flow [6], mixing and aero-acoustic. A circular jet is formed when fluid is emitted, with a given initial momentum, out of a circular orifice into a large space. As one of the generic flows of fluid mechanics, jets have been studied extensively. Spatially developing incompressible circular jet represents a free shear layer that is convectively unstable at low Reynolds numbers. In such cases, small perturbations are amplified as they propagate downstream. The stability properties of the jet flow play a fundamental role in the transition to turbulence and the formation of coherent vortex structures in a turbulent fluid [7, 8]. The ability to predict and control flow in jet is important to developing and improving designs of industrial equipment.

Frequently, the choice of independent variables is motivated by the symmetry of circular jet, then cylindrical co-ordinates are likely most appropriate to describe the governing equation. However, the choice of a particular set of independent variable might inadvertently introduce mathematically allowable but physically unrealistic terms, *e. g*., singularities. The treatment of the geometrical singularity in cylindrical and spherical co-ordinates has been a difficulty in the development of accurate finite difference and pseudo-spectral schemes for many years [9]. The use of a spectral representation is often to be preferred for the accurate solution of problems with simple geometry. Priymak and Miyazakiy [10] presented a robust numerical technique for incompressible Navier-Stokes equations in cylindrical co-ordinates. Meseguer and Trefethen [11] described a Fourier-Chebyshev Petrov-Galerkin spectral method for high accuracy computation of linearized dynamics flow in finite circular pipe in the light

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of Chapman's analysis. For hydrodynamic stability problems, the linearized Navier-Stokes equations are a general eigenvalues equation; the conditioner of the matrix usually is an ill-conditioned system arising from very-high order polynomial interpolations. Trefethen *et al.* [12] have highlighted the fact that, even when all eigenvalues of the linearized problem indicated decay, there can be transient amplification and growth in energy owing to the non-normality of the linearized operator. Theoretical approaches to extract perturbations that are efficient in triggering turbulence are presented by Biau and Bottaro [13], respectively.

For circular jet, the unbounded domain is another problem to be overcome. To construct basis function for unbounded domains, it is necessary to assume the asymptotic behavior of the approximated functions for large radius *r*. One way to treat this class of functions is the domain truncation method, which imposes artificial boundary conditions at sufficiently large radius. The method can be made more efficient if additional mappings are used, so that standard spectral basis functions such as Chebyshev polynomial can be used. Grosch and Orszag [14] investigated the exponential and algebraic mapping methods and found by numerical experiments that the algebraic mapping gives a better result than the exponential mapping in the semi-infinite field. Boyd [15] supported their result by examining the asymptotic behavior of the expansion coefficients of model functions by the method of steepest descent. Recently, Xie and Lin [16] pointed out that the algebraic mapping has some drawbacks in the axisymmetric geometry, such as clustered points near the origin, and the use of exponential mapping may treat the problem near the origin region easily, *e. g.* an extra function can be included in the basis functions to represent the far-field behavior of the expanded functions more efficiently.

In the present work, the spectral Petrov-Galerkin scheme for the numerical approximation of hydrodynamic stability equation in a circular jet is presented. The solenoidal condition introduces a linear dependence between the radial, azimuthal and axial components of the fields. The results reveal that critical Reynolds numbers with various basis functions and weight function are consistent with each other.

The mathematical formulation

We investigate the utility of mappings to solve the linear stability problems of round jet in infinite regions numerically. To expand this class of functions, we consider the exponential mapping:

$$x=\frac{1-\mathrm{e}^{-r/L}}{1+\mathrm{e}^{-r/L}} \quad r\in(-\infty,\infty) \quad \text{or} \quad r/L=\ln\frac{1+x}{1-x} \quad x\in(-1,1) \tag{1}$$

where L is the map parameter. Then the linearized Navier-Stokes equations in cylindrical co-ordinates are:

$$\begin{aligned}
-ikcu_r &= -Dp+\frac{1}{\mathrm{Re}}\left(D^2u_r+\frac{1}{r}u_r-\frac{n^2+1}{r^2}u_r-k^2u_r-\frac{2}{r^2}inu_\theta\right)-ikU_zu_r\\
-ikcu_\theta &= -\frac{in}{r}p+\frac{1}{\mathrm{Re}}\left(D^2u_\theta+\frac{1}{r}u_\theta-\frac{n^2+1}{r^2}u_\theta-k^2u_\theta+\frac{2}{r^2}inu_r\right)-ikU_zu_\theta\\
-ikcu_z &= -ikp+\frac{1}{\mathrm{Re}}\left(D^2u_z+\frac{1}{r}u_z-\frac{n^2}{r^2}u_z-k^2u_z\right)-u_rDU_z-ikU_zu_z\\
&\left(D+\frac{1}{r}\right)u_r+i\left(\frac{1}{r}nu_\theta+ku_z\right)=0
\end{aligned} \tag{2}$$

where the operator is defined as $D = (1 - x^2)(\mathrm{d}/\mathrm{d}x)/2L$, and $u_r(x)$, $u_\theta(x)$, $u_z(x)$, and $p(x)$ are the amplitudes of the corresponding disturbances, n is the azimuthal mode of the disturbance, k – the axial wavenumber of disturbance, and c (or $\beta = kc$) – the wave amplification factor. These equations are non-dimensionalised with respect to momentum thickness L^*, the jet core velocity U^*, and Reynolds number is $\mathrm{Re} = L^*U^*/\nu$. The boundary conditions for first azimuthal ($n = 1$) mode become:

$$u_r(0) + u_\theta(0) = Du_z(0) = Dp(0) = 0; \quad u_r(1) = u_\theta(1) = u_z(1) = p(1) = 0 \tag{3}$$

Solenoidal Petrov-Galerkin discretization

In order to have spectral accuracy in the numerical approximation of the eigenvalues problem, the solenoidal basis for the approximation of the perturbation vector field takes the form:

$$\mathbf{u} = \mathrm{e}^{i(kz+n\theta-kct)} \sum_{m=0}^{M} a_m^{(1)} \mathbf{w}_m^{(1)}(x) \mathbf{u}_m^{(1)}(x) + \mathrm{e}^{i(kz+n\theta-kct)} \sum_{m=0}^{M} a_m^{(2)} \mathbf{w}_m^{(2)}(x) \mathbf{u}_m^{(2)}(x) \tag{4}$$

where $\mathbf{u}_m$ belongs to the physical space and $\mathbf{w}_m$ is a solenoidal vector field belongs to the projection space. The vector fields $\mathbf{u}_m^{(1)}$ and $\mathbf{u}_m^{(2)}$ satisfy the zero divergence condition:

$$\nabla \mathrm{e}^{i(kz+n\theta-kct)} \mathbf{u}_m^{(1,2)}(x) = 0 \tag{5}$$

There is a linear dependence between the components of the vector field according to the perturbation continuum equation. Using the principle of permutation and combination, $\mathbf{u}_m$ and $\mathbf{w}_m$ have three basic forms:

– **Case I**: rendering the azimuthal free, then the physical basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} -inr^{n-1} g_m(x) \\ D[r^n g_m(x)] \\ 0 \end{pmatrix}, \quad \mathbf{u}_m^{(2)} = \begin{pmatrix} 0 \\ -ikr^n h_m(x) \\ inr^{n-1} h_m(x) \end{pmatrix} \tag{6a}$$

The projection fields are going to have the same structure as the trial fields but the functions will be modified by the Chebyshev weight $(1 - x^2)^{-1/2}$. They can be bounded if the projection velocity fields as:

$$\mathbf{w}_m^{(1)} = \frac{1}{\sqrt{1-x^2}} \begin{pmatrix} -inr^n g_m(x) \\ D[r^{n+1} g_m(x)] + r^{n+1} x g_m(x)/2L \\ 0 \end{pmatrix}, \quad \mathbf{w}_m^{(2)} = \frac{1}{\sqrt{1-x^2}} \begin{pmatrix} 0 \\ -ikr^{n+1} h_m(x) \\ inr^n h_m(x) \end{pmatrix} \tag{6b}$$

– **Case II**: rendering the axial free, the physical basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} -ikr^{n-1} g_m(x) \\ 0 \\ D[r^n g_m(x)]/r \end{pmatrix}, \quad \mathbf{u}_m^{(2)} = \begin{pmatrix} 0 \\ -ikr^n h_m(x) \\ inr^{n-1} h_m(x) \end{pmatrix} \tag{7a}$$

The corresponding projection fields is:

$$\mathbf{w}_m{}^{(1)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} -ikr^n g_m(x) \\ 0 \\ D[r^{n+1}g_m(x)]/r + r^n g_m(x)x/2L \end{pmatrix}, \quad \mathbf{w}_m{}^{(2)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} 0 \\ -ikr^{n+1}h_m(x) \\ inr^n h_m(x) \end{pmatrix} \tag{7b}$$

– **Case III**: rendering the radial free, the physical basis is:

$$\mathbf{u}_m{}^{(1)} = \begin{pmatrix} -inr^{n-1}g_m(x) \\ D[r^n g_m(x)] \\ 0 \end{pmatrix}, \quad \mathbf{u}_m{}^{(2)} = \begin{pmatrix} -ikr^{n-1}h_m(x) \\ 0 \\ D[r^n h_m(x)]/r \end{pmatrix} \tag{8a}$$

The corresponding projection fields is:

$$\mathbf{w}_m{}^{(1)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} -inr^n g_m(x) \\ D[r^{n+1}g_m(x)] + r^{n+1}xg_m(x)/2L \\ 0 \end{pmatrix},$$
$$\mathbf{w}_m{}^{(2)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} -ikr^n h_m(x) \\ 0 \\ D[r^{n+1}h_m(x)]/r + r^{n+1}h_m(x)x/2L \end{pmatrix} \tag{8b}$$

for various azimuthal mode n, except that if $k = 0$, the third component of $\mathbf{w}_m{}^{(2)}$ is replaced by $rh_m(x)$.

Substituting the spectral series in equations and projecting over the dual space carries out the Petrov-Galerkin projection scheme. This procedure leads to a discretized generalized eigenvalues problem, and the coefficient $a_m{}^{(1,2)}$ govern the temporal behavior:

$$[\mathbf{A}\,\mathbf{X}] = -ikc[\mathbf{B}\,\mathbf{X}] \tag{9}$$

where the matrices **[A]**, [**B**] and **[X]** represent:

$$[\mathbf{A}] = \begin{bmatrix} \left(\mathbf{w}_m{}^{(1)} \cdot \ell[\mathbf{u}_m{}^{(1)}]\right) & \left(\mathbf{w}_m{}^{(1)} \cdot \ell[\mathbf{u}_m{}^{(2)}]\right) \\ \left(\mathbf{w}_m{}^{(2)} \cdot \ell[\mathbf{u}_m{}^{(1)}]\right) & \left(\mathbf{w}_m{}^{(2)} \cdot \ell[\mathbf{u}_m{}^{(2)}]\right) \end{bmatrix}; \quad [\mathbf{B}] = \begin{bmatrix} \left(\mathbf{w}_m{}^{(1)} \cdot \mathbf{u}_m{}^{(1)}\right) & \left(\mathbf{w}_m{}^{(1)} \cdot \mathbf{u}_m{}^{(2)}\right) \\ \left(\mathbf{w}_m{}^{(2)} \cdot \mathbf{u}_m{}^{(1)}\right) & \left(\mathbf{w}_m{}^{(2)} \cdot \mathbf{u}_m{}^{(2)}\right) \end{bmatrix}; \quad [\mathbf{X}] = \begin{bmatrix} a_m{}^{(1)} \\ a_m{}^{(2)} \end{bmatrix} \tag{10}$$

in which ℓ stands for the linear operator of linear stability equations:

$$\ell[\cdot] = \frac{1}{\mathrm{Re}}\Delta[\cdot] - \mathbf{u}_B \cdot \nabla[\cdot] - [\cdot] \cdot \nabla \mathbf{u}_B$$

where $\mathbf{u}_B$ is the basic flow velocity vector (0, 0, U_z). The pressure term should be formally included in the operator ℓ, but it is cancelled when projecting it over **w**, that is $(\mathbf{w}, \nabla p) = 0$.

Results and discussion

The generalized eigenvalues problem in eq. (9) can be computed exactly by Gauss-Chebyshev-Lobatto quadrature formulas. In the present study, the temporal instability of round jet is considered. Hence, n is necessarily an integer, and k is real quantity while $c = c_r + ic_i$ is generally complex. The flow will be temporal unstable if the imaginary part of the complex amplification is positive. And the eigenvalues can be obtained by parametrically varying the

Reynolds number and frequency for an azimuthal wave number $n = 1$. This mode represents the most important components of circular jet flow. The eigenvalues is affected by some parameters, such as the order of Chebyshev polynomial, M; exponential map parameter, L; axial wavenumber, k; and the Reynolds numbers, Re, *etc*. According to previous study [16], $M = 100$ is far away enough for the accuracy of wave amplifications, the map parameter values with $L = 3$ represents the best compromise between the competing demands of the accuracy and the cost of computation.

For circular jet velocity profile, it is unstable for small Reynolds number. Table 1 shows the comparison of eigenvalues for various cases under supercritical conditions (Re = 40, $k = 0.469$), and theses eigenvalues are consistent generally. From the point of computation cost, the basis function in case II has an additional division operation and the basis function in case III need more differential operators than that in case I. Therefore the basis functions in case I is more preferable in numerical simulation.

Table 1. Comparison of least stable eigenvalues under supercritical conditions

	Re	k	c_r	c_i
Case I	40	0.469	–0.2603	0.0044
Case II	40	0.469	–0.2603	0.0045
Case III	40	0.469	–0.2621	0.0044

The phase velocity and amplification factor and as a function of wavenumber (k) under Re = 60 for different cases are shown in fig.1. For the $n = 1$ mode the phase velocity decreases monotonically with frequency or wavenumber. Whereas the amplification factor has a peak value in the supercritical zones in the k-Re plane when Reynolds number is given, and the corresponding wavenumber is about $k = 0.469$. It can be found that the curves of amplification factor and phase velocity as a function of wavenumber are overlapped approximately for different cases.

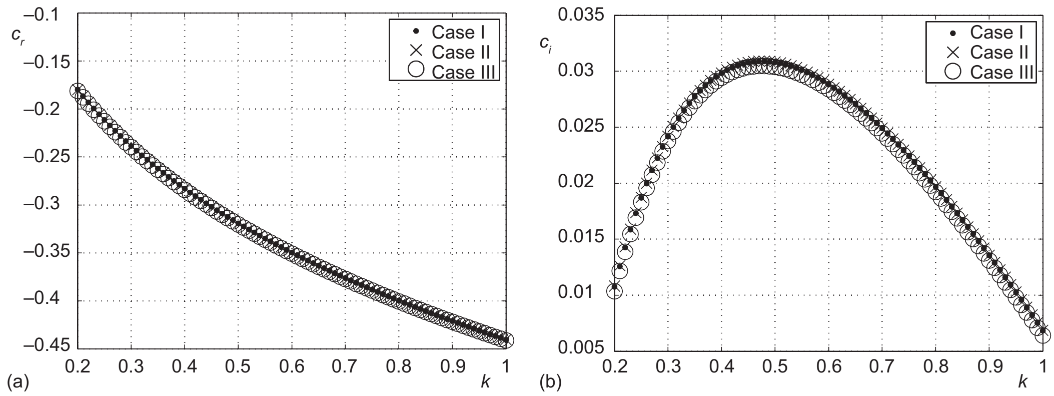

Figure 1. The phase velocity and amplification factor as a function of *k* with Re = 60

The disturbances will grow with time if $c_i > 0$ and will decay if $c_i < 0$. The neutral disturbances are then characterized by $c_i = 0$. And the neutral curves for various cases in k-Re plane and c_r-Re plane are plot in fig. 2. In k-Re plane the neutral curve ($c_i = 0$ line) separates the space into two zones: one is stable and the other is unstable. The critical Reynolds number is the point where the curve c_i(Re) becomes tangent to the $c_i = 0$ line. Generally, the neutral curves for three cases are consistent with each other. The critical Reynolds number is found to be Re = 37.64, and the corresponding wave number is $k = 0.469$ for $n = 1$ mode, and the phase velocity is $c_r = –0.253$.

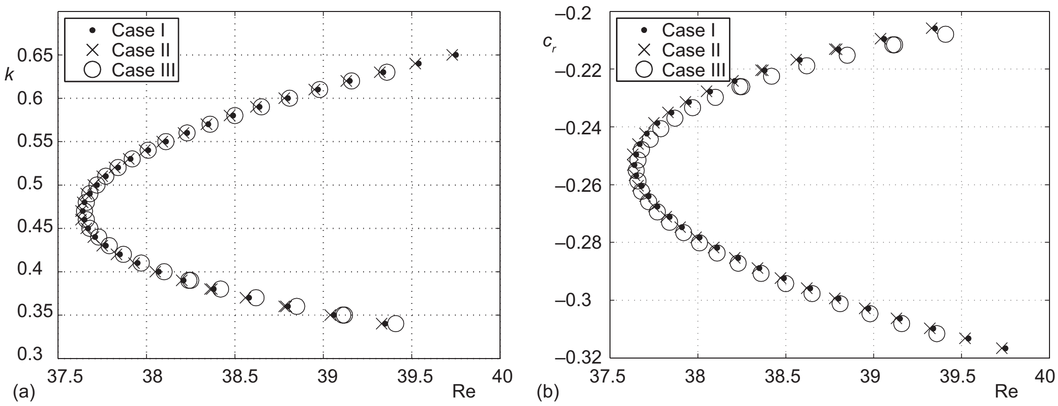

Figure 2. The neutral curve in *k*-Re plane and c_r-Re plane based on critical c_i = 0

Conclusions

The incompressible linear stability equation of round jet in cylindrical polar co-ordinates with Petrov-Galerkin spectral method is presented. Basis functions presented here are exponentially mapped Chebyshev functions. There is a linear dependence between the components of the perturbation vector field, and there are only two degrees of freedom for the perturbation continuum equation, therefor, the basis function has three basic forms, *i. e.*, rending the radial, azimuthal or axial component free. The results show that least stable eigenvalues, neutral curves and critical Reynolds numbers for various cases are consistent, but the basis function in case I is preferable for numerical computation due to its simple and concise.

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