

REDISTRIBUTION OF MASS FROM A THIN INTERLAYER BETWEEN TWO THICK DISSIMILAR MEDIA 1-D diffusion problem with a non-local condition

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Diffusion problem with a specification of considering liquid redistribution from a thin interlayer between two semi-infinite media in contact is developed. The basic approach involves an integral approach defining finite depths of penetration of the diffusant into the media and fractional half-time derivative of the boundary (at the interface) concentration. The approach is straightforward and avoids cumbersome calculations based on the idea to develop entire domain (for each of the contacting bodies) solutions. The results are compared to classical solutions, when they exist.

Key words: *diffusion, specification of mass, integral method, fractional time semi-derivatives*

Introduction

The diffusion problems with specification of energy or mass are non-linear ones due to the imposed non-local conditions and encountered in many practical situations. The present article addresses a parabolic problem with a specification of mass with in case of two interaction media (bodies) in contact. This problem, in accordance with the literature background, has been considered in a short communication [1] about 30 years ago and practically undeveloped so far. Now, we propose a new solution based on an integral approach and involving fractional semi-derivatives

Problem background

The mass (liquid) redistribution in virgin media at constant or variable conditions imposed by the penetrating liquid at the boundary interface is a problem relevant to many technological processes such as bonding of paper [2], wood [3], bonding of metals [4, 5], dental medicine [6, 7], bone repairing [8], etc. Commonly during the process the filler liquid is imposed at the boundary $x = 0$ as a limited amount of substance (a momentary source) in the form of a thin layer with thickness negligible with the respect to the scales of the contacting bodies which have to be bonded.

There exist two approaches in modelling the liquid penetration process, namely:

- (a) Transient diffusion with constant concentration at the boundary $x = 0$ which is commonly used in modeling the transient liquid phase bonding of metal alloys where a thick layer of a bonding metal (which consequently melts and penetrates) is used [4, 5, 9]. These models are commonly solved analytically as Stefan problems [10].

(b) Transient diffusion with a specification of mass, which means a non-local (integral) condition relevant to an instantaneous release of a limited amount of mass from a thin layer at $x = 0$ and decaying in time [1, 11]. The problem is relevant to paper bonding, powder sintering, and many other technological process and developed in this article.

Additional to its practical importance, the problem is, to some extent, a mathematical challenge referring to case which are mainly solved numerically [12, 13] or by complex analytical techniques [1, 14]. In the dominating cases the problem of interest is a semi-infinite medium [1, 12] or areas of limited size [14] with specification of mass (energy). The problem is also relevant to transient heat conduction where short limited pulse of energy is released at the boundary by a thin high-conductivity layer.

The present work presents a new solution relevant to redistribution of mass from a thin layer between two dissimilar bodies in contact under an integral (non-local) condition. It is shown that these are separate diffusion processes with a time-dependent condition at the boundary ($x = 0$) which depends on the diffusivities of the bodies. The problem with such a formulation is solved by applying integral approach and time-fractional semi-derivatives [15].

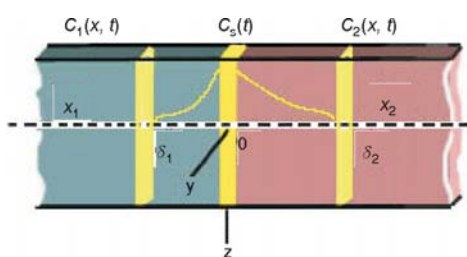


Figure 1. Schematic view of the physical problem of a mass redistribution between contacting bodies

Problem formulation

The governing equation describing the diffusion process of mass redistribution between the contacting bodies (see fig. 1) is:

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0 \quad (1a)$$

The initial conditions refer to the entire areas of the bodies are:

$$C(x, 0)_{x \neq 0} = 0 \quad (1b)$$

as well as the far-field condition of virgin media, namely:

$$C(x, t)_{x \rightarrow -\infty} = C(x, t)_{x \rightarrow +\infty} = 0 \quad (1c)$$

The transport characteristics (diffusivities) of both bodies are:

$$D = D_1(x < 0), \quad D = D_2(x > 0) \quad (1d)$$

with a condition at the interface $x = 0$:

$$C_1(x, t)_{x \rightarrow 0} = C_2(x, t)_{x \rightarrow +0} = 0 \quad (1e)$$

The integral condition is described in general as

$$\int_{-\infty}^{+\infty} C(x, t) dx = M(t), \quad t > 0, \quad \text{or} \quad \int_{-\infty}^{-0} C_1(x, t) dx + \int_{+0}^{+\infty} C_2(x, t) dx = M(t), \quad t > 0 \quad (2a,b)$$

where $M(t)$ represents the amount of mass released by thin layer.

In general, the diffusion problems with integral conditions like (2a) are solved numerically [13]. Because of that, the present article focuses on an explicit analytical solution which can be easily developed and applied to various forms of the source function $M(t)$ defined by the integral condition; the most interesting of them are the "constant source" and the "infinite source" [16].

• **Note 1:** About the units used in the model

In order to make clear all further expressions, we have to address the dimensions of the terms in the equations of the model. For better presentation of the solution and the comments on it, we especially avoid the non-dimensionalization, which, to some extent, in the case of fractional equations is impossible. First of all, if SI units are used, then the dimensions of $M(t)$ is kg or kg/s. Hence, if we have $M(t)$ in kg, then $C(x, t)$ has a dimension kg/m because we have 1-D problem. In this context, when $x = 0$, the surface concentration $C(x, t)_{x \rightarrow -0} = C(x, t)_{x \rightarrow +0} = C_s(t)$ has a dimensions kg/s (see further the mass balance expressed in the form of eqs. (25a-d). Further, because the terms \sqrt{D} emerge in many of the expressions in this work, we have \sqrt{D} [$\text{ms}^{-1/2}$] because D [m^2s^{-1}].

Solution

The mass balance

Let us assume that there are finite penetration depths of the liquid in both media *i. e.* $\delta_1(x < 0)$ and $\delta_2(x > 0)$. This leads to a new form of eq. (2b), namely:

$$\int_{\delta_1}^0 C(x_1, t) dx + \int_0^{\delta_2} C(x_2, t) dx = M(t), \quad t > 0 \quad (3a)$$

Moreover, with this assumption the boundary conditions (1c) are replaced by:

$$C_1(\delta_1, t) = C_2(\delta_2, t) = 0 \quad (3b)$$

and

$$\frac{\partial C_1(\delta_1, t)}{\partial x} = \frac{\partial C_2(\delta_2, t)}{\partial x} = 0 \quad (3c)$$

The integration of the governing equations for each body with respect to the space co-ordinates x_1 and x_2 yields:

$$\int_{\delta_1}^0 \frac{\partial C_1(x_1, t)}{\partial t} dx = -D_1 \left. \frac{\partial C_1}{\partial x_1} \right|_{x=-0} \quad (4a)$$

$$\int_0^{\delta_2} \frac{\partial C_2(x_2, t)}{\partial t} dx = -D_2 \left. \frac{\partial C_2}{\partial x_2} \right|_{x=+0} \quad (4b)$$

Now, applying consequently the Leibniz rule to the LHS of eqs. (4a) and (4b) and then a summation, we get:

$$\frac{\partial}{\partial t} \left[\int_{\delta_1}^0 C_1(x_1, t) dx + \int_0^{\delta_2} C_2(x_2, t) dx \right] = \frac{\partial M(t)}{\partial t} = -D_1 \left. \frac{\partial C_1}{\partial x_1} \right|_{x=-0} - D_2 \left. \frac{\partial C_2}{\partial x_2} \right|_{x=+0} \quad (5)$$

The result (5) simply expresses the physical process that the variation in the quantity of the liquid in time is balanced by the fluxes into both media. In this context, we have simple diffusion problems in each of the media with a time-varying boundary condition at $x \rightarrow -0$ and $x \rightarrow +0$, *i. e.* $x = 0$. This formulates a new problem defining the variation of the surface (boundary) concentration $C_s(t)$.

The surface (boundary) concentration at $x = 0$

Approach 1

Splitting the parabolic eq. (1a) using the exponential rule in terms of Riemann-Liouville (RL) fractional semi-derivative we get two fractional half-time subdiffusion equations [15, 17], namely:

$$\frac{\partial^{1/2} C_1(0, t)}{\partial t^{1/2}} = -\sqrt{D_1} \frac{\partial C_1}{\partial x} \Big|_{x=-0} \quad (6a)$$

$$\frac{\partial^{1/2} C_2(0, t)}{\partial t^{1/2}} = -\sqrt{D_2} \frac{\partial C_2}{\partial x} \Big|_{x=+0} \quad (6b)$$

Recall, we have zero initial conditions that allows applications of RL derivatives and those of the Caputo sense, too.

Now, taking into account the boundary condition (1e), and denoting $C(x, t)_{x \rightarrow 0} = C(x, t)_{x \rightarrow +0} = C_s(0, t)$ we may express (6a, b) as:

$$D_t^{1/2} C_s = -\sqrt{D_1} \frac{\partial C_s}{\partial x} \Big|_{x \rightarrow -0} \Rightarrow -\frac{\partial C_s}{\partial x} \Big|_{x \rightarrow -0} = \frac{1}{\sqrt{D_1}} D_t^{1/2} C_s \quad (7a)$$

$$D_t^{1/2} C_s = -\sqrt{D_2} \frac{\partial C_s}{\partial x} \Big|_{x \rightarrow +0} \Rightarrow -\frac{\partial C_s}{\partial x} \Big|_{x \rightarrow +0} = \frac{1}{\sqrt{D_2}} D_t^{1/2} C_s \quad (7b)$$

Here $D_t^{1/2} C_s$ denotes a time-fractional RL derivative (operator) $\partial^{1/2}/t^{1/2}$, where [15]:

$$\frac{\partial^{1/2} C(x, t)}{\partial_+ t^{1/2}} = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t \frac{C(x, u)}{\sqrt{t-u}} du \quad (8a)$$

In this context, the time-fractional integral $\partial^{-1/2}/t^{-1/2}$ [15] is:

$$D_t^{-1/2} = \frac{1}{\Gamma(1/2)} \int_0^t \frac{C(x, u)}{\sqrt{t-u}} du \quad (8b)$$

In eqs. (8a) and (8b), u is a dummy variable and $\Gamma(1/2) = \sqrt{\pi}$.

Moreover, we can use further the notation $\partial C_s / \partial x$ for the derivatives $\partial C_s / \partial x|_{x \rightarrow -0}$ and $\partial C_s / \partial x|_{x \rightarrow +0}$.

Hence the mass balance eq. (5) can be expressed as:

$$\frac{\partial M(t)}{\partial t} = -(D_1 + D_2) \frac{\partial C_s}{\partial x} \quad (9a)$$

with the expressions about $\partial C_s / \partial x$ from (7a, b) and substitution in (9a) we get:

$$\frac{1}{(\sqrt{D_1} + \sqrt{D_2})} \frac{\partial M(t)}{\partial t} = D_t^{1/2} C_s \quad (9b)$$

Recall, $D_t^{-1} f = \int_0^t f dt$ and applying the operator D^{-1} (RL sense) to both sides of eq. (9b) we get an equation about C_s :

$$D_t^{-1/2} C_s = \frac{M(t)}{\sqrt{D_1} + \sqrt{D_2}} \quad (10a)$$

Then, the solution is:

$$C_s(t) = D_t^{1/2} \left(\frac{M(t)}{\sqrt{D_1} + \sqrt{D_2}} \right) \quad (10b)$$

Approach 2

Let us consider other way to develop the result of (10b). We will demonstrate how this can be done avoiding the use of the notation $C(x, t)_{x \rightarrow -0} = C(x, t)_{x \rightarrow +0} = C_s(0, t)$.

The integral condition (2b) can be integrated twice with respect the space co-ordinate that yields:

$$\frac{\partial}{\partial t} \left\{ \int_{-\delta_1}^{\delta_2} \left[\int_{\delta_1}^0 C_1(x_1, t) dx + \int_0^{\delta_1} C_2(x_2, t) dx \right] dx \right\} = (\delta_1 + \delta_2) \frac{\partial M(t)}{\partial t} = -C_s (D_1 - D_2) \quad (11a)$$

After application of the same approach to the time-fractional sub eqs. (7a) and (7b), and a consequent summation we get:

$$(\delta_1 + \delta_2) D^{1/2} C_s = -C_s (\sqrt{D_1} - \sqrt{D_2}) \quad (11b)$$

Eliminating the sum $\delta_1 + \delta_2$ from eqs. (11a) and (11b) we have:

$$D^{1/2} C_s = \frac{1}{(\sqrt{D_1} + \sqrt{D_2})} \frac{\partial M(t)}{\partial t} \quad (11c)$$

Hence, we got the result of (9b) by a double integration which is a common technique in the heat-balance integral method [18, 19], thus avoiding errors in approximation of the space derivative at $x = 0$. Approach 2 allows explicitly demonstrating the physical sound of the assumption of the final penetration depths expressed by the condition (3b). Recall, the integration with respect to the space co-ordinate in the Approach 1 can be performed in $[-\infty, \infty]$ instead $[-\delta_1, \delta_2]$: the result will be the same as in (9b) and (11c).

• **Note 2:** About the correct relation of the flux to the fractional time derivative

The correct relationship between the flux and the time semi-derivative for the diffusion equation [15] is (the left-side medium equation, for example):

$$-\frac{\partial C_1(x, t)}{\partial x} = \frac{1}{\sqrt{D_1}} \left[D^{1/2} C_1(x, t) - \frac{C_1(x, 0)}{\sqrt{\pi t}} \right] \quad (12a)$$

However, due the initial condition (1b) we have:

$$-\frac{\partial C_1(x, t)}{\partial x} = \frac{1}{\sqrt{D_1}} D^{1/2} C_1(x, t) \quad (12b)$$

This relationship was used in the development of the problem and will be utilized further in the text.

Some cases in relation to the function $M(t)$

The solutions (9b) and (11c) are straightforward and depend on type of the function $M(t)$. In this context, four types of $M(t)$ functions are exemplified briefly, namely:

(a) Instantaneous load of limited amount of liquid $M_0 = \text{const.}$ at $t = 0, x = 0$

In this case the solution is straightforward:

$$C_s(t) = D^{1/2} \left(\frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} \right) = \frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} \quad (13a)$$

(b) Ramp (linear rise) load in $M(t)$

With a ramp function $M(t) = k_M t$ we have $\partial M(t)/\partial t = k_M = \text{const.}$ and from eq. (9b) we get the solution:

$$D^{1/2} C_s = \frac{k_M}{\sqrt{D_1} + \sqrt{D_2}} \Rightarrow C_s = \frac{2k_M}{\sqrt{D_1} + \sqrt{D_2}} \sqrt{\frac{t}{\pi}} \quad (13b)$$

(c) Exponential increase, *i. e.* $M(t) = \exp(\alpha t)$.

From eq. (10b) we have:

$$C_s(t) = D^{1/2} \left(\frac{\exp(\alpha t)}{\sqrt{D_1} + \sqrt{D_2}} \right) = \frac{1}{\sqrt{D_1} + \sqrt{D_2}} \left[\frac{1}{\sqrt{\pi t}} + \sqrt{\alpha} e^{\alpha t} \text{erf}(\sqrt{\alpha t}) \right] \quad (14a)$$

(d) Exponential decay, *i. e.* $M(t) = \exp(-\beta t)$

$$C_s(t) = D^{1/2} \left(\frac{\exp(-\beta t)}{\sqrt{D_1} + \sqrt{D_2}} \right) = \frac{1}{\sqrt{D_1} + \sqrt{D_2}} \left[\frac{1}{\sqrt{\pi t}} - \frac{2}{\sqrt{\pi}} \text{daw}(\sqrt{\beta t}) \right] \quad (14b)$$

because

$$\frac{\partial^{1/2}}{\partial t^{1/2}} [\exp(-t)] = \frac{1}{\sqrt{\pi t}} - \frac{2}{\sqrt{\pi}} \text{daw}(\sqrt{t}) \quad (14c)$$

and

$$\frac{\partial^{1/2}}{\partial t^{1/2}} [f(\beta t)] = \sqrt{\beta} \frac{\partial^{1/2}}{\partial (\beta t)^{1/2}} f(\beta t) \quad (14d)$$

where $\text{daw}(\sqrt{t}) = (1/2)\sqrt{\pi} \exp(-t)$ is the Dawson function [15].

These numerical examples only show what types of time-dependent boundary conditions emerge when the integral condition (2b) varies. Some simple cases will be resolved next by two classical approaches involving the Duhamel convolution integral [20] and Laplace transforms.

Concentration profiles

General approach by the Duhamel integral

Now, we know the concentration at $x=0$ and the solution for each of two bodies can be expressed through the Duhamel convolution integral, namely

$$C_1(x, t) = \int_0^t C_s(\tau) \frac{\partial S_1(x, t - \tau)}{\partial t} d\tau \quad (15a)$$

$$C_2(x, t) = \int_0^t C_s(\tau) \frac{\partial S_2(x, t - \tau)}{\partial t} d\tau \quad (15b)$$

The application of the Duhamel integrals is valid due to the zero initial conditions (1b) and (1c). The functions $S_1(x, t)$ and $S_2(x, t)$ are solutions obtained when there is a unit step change $C_s^0 = 1$ at $x = 0$.

With $S(t) = 1 - \text{erf}(-x/\sqrt{4Dt}) = \text{erfc}(-x/\sqrt{4Dt})$, we have :

$$C_1(x, t) = \int_0^t \left[\frac{1}{\sqrt{4\pi D_1(t-\tau)}} e^{-\frac{x^2}{4D_1(t-\tau)}} \right] C_s(\tau) d\tau \quad (16)$$

In case of a limited amount of liquid $M_0 = \text{const.}$ at $t = 0, x = 0$, from eqs. (12a) and (15a), we have the classic solution (see example 5-2 in [20]) for $C_s(t)$:

$$C_1(x, t) = \frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} e^{-\frac{x^2}{4D_1 t}}, \quad C_2(x, t) = \frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} e^{-\frac{x^2}{4D_2 t}} \quad (17a,b)$$

Recall, the negative sign in exponent of eqs. (17a,b) does not alter due to the fact that left side medium concentration (C_1) varies along the axis $(0, -\infty)$ because after the determination of the surface concentration C_s the problem splits into two independent diffusion problems in semi-infinite media. If the sign in expression (7a) alters to a positive this yields a growing with the space variable concentration which unphysical. Hence, after the determination of C_s we have two-classic transient diffusion problems with time-dependent boundary condition at $x = 0$.

With this simple example we only demonstrate that the new solution using fractional semi-derivatives results in a well-known classical result. The analysis, in the next section, however, will avoid this approach due the complexity in the direct evaluation of the Duhamel integral when the function $M(t)$ is different from $M_0 = \text{const.}$ We stress the attention on short-time solutions as it is explained next.

Laplace transforms and short-time solutions

Laplace transforms of the time convolution

In order to avoid the cumbersome calculations of the Duhamel integrals emerging when different functions of $M(t)$ are defined we use the approach with the Laplace transform, remembering that the solution is a time-convolution represented in two equivalent forms as:

$$C(x, t) = \frac{\partial C_s(t)}{\partial t} S(x, t) \quad \text{or} \quad C(x, t) = C_s(t) \frac{\partial S(x, t)}{\partial t} \quad (18a)$$

The Laplace transform (denoting for simplicity $\dot{C}_s(t) = \partial C_s(t)/\partial t$ of the time convolution is [15]:

$$L[C(x, t)] = L[\dot{C}_s(t)S(x, t)] = C_L(x, s) = \dot{C}_s(s)S(x, s) \quad (18b)$$

We will continue further with the case of a ramp function $M(t) = k_M t$ satisfying the integral condition (2b). In the case of a ramp function $M(t) = k_M t$, we have [15] a unit step solution expressed by the error-function and its image is ($L(\bullet)$ is the Laplace operator):

$$S(x, t) = \text{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right) \Rightarrow L[S(x, t)] = \frac{1}{s} \exp\left[-\left(\frac{x^2}{D}\right)s\right] \quad (19a)$$

with $C_s = 2k_M/(\sqrt{D_1} + \sqrt{D_2})\sqrt{t/\pi}$ the Laplace transform of its time-derivative is:

$$\dot{C}_s(t) = \frac{2k_M}{\sqrt{D_1} + \sqrt{D_2}} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \Rightarrow L[\dot{C}_s(t)] = \frac{2k_M}{\sqrt{D_1} + \sqrt{D_2}} \frac{1}{\sqrt{s}} \quad (19b)$$

Hence, the image $C_L(x, s)$ is (left-side medium, for example):

$$C_L(x, s) = \frac{2k_M}{\sqrt{D_1} + \sqrt{D_2}} \frac{1}{s\sqrt{s}} \exp\left[-\left(\frac{x^2}{D_1}\right)s\right] \equiv \frac{1}{s\sqrt{s}} \exp(-as); \quad a = \frac{x^2}{D_1} \quad (19c)$$

The inverse Laplace transform (when $a > 0$) of the image $C_L(x, s)$ expressed by eq. (19c) [15] is:

$$L^{-1}\left[\frac{1}{s\sqrt{s}}\exp(-as)\right] = 2\frac{\sqrt{t}}{\sqrt{\pi}}\exp\left(-\frac{a}{4t}\right) - \sqrt{a}\operatorname{erfc}\left(\frac{\sqrt{a}}{2\sqrt{t}}\right) \quad (20a)$$

Hence, the concentration profile (the left-side medium) is:

$$C_1(x, t) = \frac{2k_M}{\sqrt{D_1} + \sqrt{D_2}} \left[2\frac{\sqrt{t}}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4D_1t}\right) - \sqrt{\frac{x^2}{D_1}}\operatorname{erfc}\left(\frac{\sqrt{x}}{2\sqrt{D_1t}}\right) \right] \quad (20b)$$

Short-time solutions

However, in many cases such simple inverse transform could be impossible, and we might be interested in short time-solutions taking into account that the problem developed is a model of many short-time technological processes as it was mentioned at the beginning.

If we look for short-time solutions, then for seek of simplicity assuming $a = 1$ in eq. (19c) (in fact $a = x^2/D_1$ is the time-scale of the process) and applying the Laplace transform we get the image:

$$C_L(a=1, s) = \frac{\exp(-\sqrt{s})}{s\sqrt{s}} \quad (21a)$$

The asymptotic expansion (performed by Maple) of $C_L(a=1, s)$ is:

$$C_L(a=1, s)_{s \rightarrow \infty} \approx \frac{1}{s\sqrt{s}}\exp\left(\frac{1}{s}\right) \quad (21b)$$

Then, taking into account that $s \rightarrow \infty$ means $t \rightarrow 0$ [21] by the inverse transform of eq. (21b), we get a short time solution in case of a ramp load $M(t) = k_M t$, namely:

$$C(a=1, t) = 2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{1}{4t}\right) - \operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right) \quad (21c)$$

Integral characteristic of the process and related parameters

Mass redistribution between the media

From a practical point of view it is quite important to know the amounts of liquid “absorbed” by the media as functions of the time. This integral characteristic of the process will be determined next in two cases.

(a) Final redistribution of mass at $t \rightarrow \infty$

First, integrating the governing equations of each of medium (see eqs.4a, b) with respect to the space co-ordinate and then applying the Leibniz rule, we get:

$$\frac{d}{dt} M_1(t) = -D_1 \left. \frac{\partial C_1}{\partial x_1} \right|_{x=-0} \quad (22a)$$

$$\frac{d}{dt} M_2(t) = -D_2 \left. \frac{\partial C_2}{\partial x_2} \right|_{x=+0} \quad (22b)$$

$$M_1(t) + M_2(t) = M(t) \quad (22c)$$

On the other hand, the half-time fractional relationships at $x = 0$ presented by eqs. (6a) and (6b) allow (22a) and (22b) to be expressed as (the left-side medium, for example):

$$\frac{d}{dt} M_1(t) = \sqrt{D_1} (D_1^{1/2} C_s) \quad (23a)$$

Now, with eq. (10a) we read:

$$M_1(t) = \sqrt{D_1} D_1^{1/2} C_s \quad (23b)$$

Therefore, with $M(t) = M_0$ (see 10a) and taking into account that $D_1^{1/2}(1/\sqrt{t}) = \sqrt{\pi}$, we read:

$$M_1(t) = \frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}} M_0 \quad (23c)$$

Similarly, for the right-side medium, we have:

$$M_2(t) = \frac{\sqrt{D_2}}{(\sqrt{D_1} + \sqrt{D_2})} M_0 \quad (23d)$$

On the other hand, the exact concentration profile in this case (left-side medium) is presented by eq. (17a). The integration with respect to the space co-ordinate defines the mass accumulated by the medium $M_1(t)$, namely:

$$\int_0^\infty C_1(x, t) dx = \int_0^\infty \left[\frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} e^{-\frac{x^2}{4D_1 t}} \right] dx \Rightarrow \left[\frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} \right] \cdot \left[\left(\frac{1}{2} \sqrt{\pi} \sqrt{D_1 t} \right) \operatorname{erf} \left(\frac{x}{\sqrt{D_1 t}} \right) \right]_0^\infty \quad (24a)$$

$$M_1(t) = \frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}} M_0 \quad (24b)$$

Hence, the result is the same as that eq. (see 23c) obtained by the integral approach and the fractional semi-derivative eqs. (6a, b). The relationships (23c) and (23d), however, do not account the time and, in fact, are valid for the final (after a long time) redistribution if the liquid between the media in contact

(b) Temporal redistribution of mass

Let us consider, the fact that the amount of liquid redistributed between the bodies for a given time t_{crit} is:

$$M_{\text{red}}(t_{\text{crit}}) = \int_0^{t_{\text{crit}}} C_s(t) dt \quad (25a)$$

Recall, the dimension of $C_s(t)$ is kg/s. Therefore, we have an integral relation:

$$\int_{-\infty}^0 C_1(x, t) dx + \int_0^{+\infty} C_2(x, t) dx = M_{\text{red}}(t_{\text{crit}}), \quad 0 < t \leq t_{\text{crit}} \quad (25b)$$

Then, the total mass balance at any time t is:

$$\int_{-\infty}^0 C_1(x, t) dx + \int_0^{+\infty} C_2(x, t) dx + C_s(t) = M(t), \quad M_{\text{red}}(t) = M(t) - C_s(t) \quad (25c, d)$$

Recall, each term in eq. (26c) has a dimension kg/s, *i. e.* $M_1(t) = \int_{-\infty}^0 C_1(x, t) dx$ is in kg/s.

In the case of $M(t) = M_0$, the redistributed mass at a given time t through eq. (12a) and (25a) at $x = 0$ (*i. e.* removed from the thin interlayer and absorbed by the contacting media) is:

$$M_{\text{red}}(t) = M_0 \left(1 - \frac{1}{\sqrt{\pi t}} \frac{1}{\sqrt{D_1} + \sqrt{D_2}} \right) \quad (26)$$

The relationship (26) is dimensionally homogeneous because the second term in the brackets has a dimension 1/m (see Note 1).

Therefore at any time $t > 0$ the value of $M_0 - C_s(t)$ should replace M_0 in eqs. (23c) and (23d), because the current concentration C_s (at any time t) and its gradients to both sides of the interface $x = 0$ is related to its time-fractional derivatives – see eqs. (6a) and (6b), and the integral mass balance (eq. 5), as well. Hence, the mass accumulated by the left-side medium, for example, is:

$$M_1(t) = \frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}} M_{\text{red}}(t) \Rightarrow M_1(t) = \frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}} M_0 \left(1 - \frac{1}{\sqrt{\pi t}} \frac{1}{\sqrt{D_1} + \sqrt{D_2}} \right) \quad (27a, b)$$

$$M_1(t \rightarrow \infty) = \frac{\sqrt{D_1}}{\sqrt{D_1} + \sqrt{D_2}} M_0 \quad (27c)$$

Hence, with eq. (27c) we got the result of (24b).

Penetration depths

If we chose the depth δ_1 defined by a line demarcating the front of the liquid penetration into the medium and assume that at $x = \delta_1$ the concentration $C_1(\delta_1, t)$ is $C_1(\delta_1, t) = \varepsilon C$ where $\varepsilon \ll 1$. In this way, we avoid directly to introduce $C_1(\delta_1, t)$ in eq. (17a). This approach in the case $M(t) = M_0 = \text{const.}$ leads to:

$$\varepsilon C_s = \frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} e^{-\frac{\delta_1^2}{4D_1 t}} \Rightarrow \delta_1^2 = \ln \left[\frac{1}{\varepsilon C_s} \frac{1}{\sqrt{\pi t}} \frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} \right] 4D_1 t \quad (28a)$$

The concentration C_s is defined by eq. (12a) and then, we get:

$$\delta_1 = 2 \sqrt{\ln \left(\frac{1}{\varepsilon} \right) \sqrt{D_1 t}} \quad (29b)$$

Hence, for example with $\varepsilon = 0.01$ the penetration depth is $\delta_{1(\varepsilon=0.01)} \approx 4.029 \sqrt{D_1 t}$, while with $\varepsilon = 0.001$ we have $\delta_{1(\varepsilon=0.001)} \approx 5.25 \sqrt{D_1 t}$.

However, this is the simplest case allowing the complete analytical solution to be used, but when the exact solution is impossible to be developed, other approach should be applied.

Let us consider eq. (22a) and integrate it again with respect to the space co-ordinates, namely:

$$\int_0^{\delta_1} \frac{d}{dt} M_1(t) dx = \int_0^{\delta_1} \left(-D_1 \frac{\partial C_1}{\partial x_1} \Big|_{x=0} \right) dx \Rightarrow \frac{d}{dt} \int_0^{\delta_1} M_1(t) dx = D_1 C_s \Rightarrow \frac{d}{dt} [\delta_1 M_1(t)] = D_1 C_s \quad (30a)$$

Hence, with the initial condition $\delta_1(t=0) = 0$ we have:

$$\delta_1 M_1(t) = C_s t \Rightarrow \delta_1(t) = \left[\frac{D_1 C_s}{M_1(t)} \right] t \quad (30b)$$

In the simplest case $M(t) = M_0 = \text{const.}$, the penetration depth $\delta_1(t)$ taking in account eqs. (23b) or (24b) and (12a) is:

$$\delta_1(t) = \frac{\sqrt{D_1 t}}{\sqrt{\pi}} \approx 0.564 \sqrt{D_1 t} \quad (30c)$$

The differences in the pre-factor of the length scale $\sqrt{D_1 t}$ in eqs. (25b) and (30c) is due to the differences in the approaches used to estimate the penetration depth. In the former case we replace the boundary $x \rightarrow -\infty$ by $x = \delta_1$ at which $C_1(\delta_1, t) = \varepsilon C_s$ and use the exact solution. The second approach expressed by eqs. (26), (4a, b), and (5), and suggests that the function $C_1(x, t)$ satisfies the governing eq. (1a) in average, which is the basic principle of the integral-balance method [18]. The differences in the fronts of the penetration layers developed by the exact solution and the integral-balance method are well analyzed and illustrated in [22].

Critical times

The time-dependent liquid penetration and redistribution of mass can be controlled by some critical times:

- (a) A critical time defining a limit in the concentration C_s at $x = 0$.

This condition comes from physical constrains of the process where after a certain period of time subsequent to the application of the liquid at the interface between the two media a polymerization of the adhesive, for example, begins.

We will illustrate this situation with two examples:

- a limit from below of the concentration C_s , *i. e.* $C_{s-\text{min}}$ in time which is related to the case of the instantaneous limited amount of liquid in $M(t) = M_0 = \text{const.}$ In this case, from (12a) we have $t_c(C_{s-\text{min}})$ as

$$t_c(C_{s-\text{min}}) = \frac{1}{\pi} \frac{1}{(C_{s-\text{min}})^2} \left(\frac{M_0}{\sqrt{D_1} + \sqrt{D_2}} \right)^2 \quad (31)$$

This critical time, consequently defines the amounts $M_1(t)$ and $M_2(t)$ through the relationships (23c) and (23d).

- a limit of C_s from above, which implies a growing in time $M(t)$.

With the simplest case of a ramp load $M(t) = k_M t$ the limit ($C_{s-\text{max.}}$) defines $t_c(C_{s-\text{max.}})$ through eq. (13a) as:

$$t_c(C_{s-\text{max}}) = \pi \left[\frac{(C_{s-\text{max}})(\sqrt{D_1} + \sqrt{D_2})}{2k_M} \right]^2 \quad (32)$$

Then, $M_1(t)$ and $M_2(t)$ through the relationships (23c) and (23d) can be estimated, and all related characteristics of the process, as well.

- (b) A critical time defined by a limit of the amount of liquid absorbed by the medium.

In this case, we have $M_1^{\text{crit}}(t)$ – similarly with $M_2^{\text{crit}}(t)$. Then, from eq. (27b) and some algebra we get:

$$t_1^{\text{crit}} = \frac{1}{\pi} \left[\left(1 - \frac{M_1^{\text{crit}}}{M_0} \frac{\sqrt{D_1} + \sqrt{D_2}}{\sqrt{D_1}} \right) (\sqrt{D_1} + \sqrt{D_2}) \right]^2 \quad (33)$$

Brief conclusions and outlines

The present communication shows an application of the fractional calculus to a solution of practical problem commonly solved either analytically [1] or numerically [11-14]. The approach involves the power of the fractional calculus to solve problems where both the function and its derivative should be determined simultaneously at the interfaces [23-25] without development of entire domain solutions. The successful application of this approach allowed in the case of the present problem to define the interface time-dependent concentration $C_s(t)$ much easier than in the complicated and unclear analytical solution developed in [1]. In accordance with the literature available such analytical solution has not been developed so far.

The main problem emerging in the solution of this problem is the non-linearity imposed by the non-local integral solution. The common approach used in solutions of such problems [11-14] is a differentiation of the integral condition with respect to the time, then applying the Leibniz rule, thus transforming the non-local condition into a modified Neumann condition (see for details ref. [13] and the analysis therein). To some extent, focusing only to the final results, the application of the integral approach used in this work provides a similar result, *i. e.* relates the time derivative of the integral condition to the gradient at $x = 0$. However, the apparent similarities are only up to this moment, because the consequent application of time-fractional semi-derivatives directly relates the surface flux (space derivatives) at $x = 0$ to the surface concentration $C_s(t)$ without development of the entire domain solutions. Hence, we convert the initial problem into a transient diffusion (for each of the contacting media) with a time-dependent condition $C_s(t)$ at $x = 0$. The next steps are well-known: either solutions by the convolution integral in case of simple functions $C_s(t)$ or approximate solutions by the Laplace transform, or some numerical methods.

The main contribution of the present paper is the approach to successful determination of the interface concentration (in the interlayer) when the two media interact and affect the redistribution of the diffusant. The case of a single medium will be more easily understood [11-14] from a physical point of view even though the approach demonstrated here is straightforward applicable to it. The simple relation between the concentrations $C_s(t)$ and the integral condition $M(t)$ through the fractional time semi-derivative allows easily defining the time-dependent boundary condition at $x = 0$.

All other problems solved here as the amounts of mass redistributed and the critical time are in fact, the practical outcomes of the problem solved. Once again, the simple relations developed by the integral approach and the fractional semi-derivatives easily resulted in simple and physically sound relationships.

Nomenclature

$C(x, t)$	– concentration (see also the comments in Note 1), [kgm^{-1}]	$M(t)$	– amount mass defined by the non-local integral condition, [kg] or [kgs^{-1}]
$C_L(x, s)$	– Laplace transform of $C(x, t)$	M_1, M_2	– amounts of masses absorbed by the contacting media, [kg] or [kgs^{-1}]
$C_s(t)$	– the concentration at the interface $x = 0$, [kgs^{-1}]	M_0	– amount mass defined by the non-local integral condition in case of instantaneous load at the interface $x = 0$, [kg]
C_{s-min}	– limit from below of the interface concentration, [kgs^{-1}]	$M_{red}(t)$	– amount of mass redistributed from the onset of the process up to the time t [kg], or [kgs^{-1}]
C_{s-max}	– limit from above of the interface concentration, [kgs^{-1}]		
D	– diffusivity, [m^2s^{-1}]		
k_M	– pre-factor in the ramp mass load function $M(t)$, [s^{-1}],		

M_1^{crit}	– critical amount absorbed by the medium, see eq. (33), [kg]	ε	– small parameter, [–]
t_1^{crit}	– critical time defined by a restriction on the amount absorbed by the medium, see eq. (33)	τ	– dummy variable in the convolution integral
$S(x, t)$	– fundamental solution of the transient diffusion problem, $C(x, t)$ developed by an exact method	Subscripts	
s	– variable after the Laplace transform	1	– left-side medium
t	– time, [s]	2	– right-side medium
t_c	– critical time, [s]	Special symbols	
x	– space co-ordinate, [m]	$D_t^{1/2}$	– time-fractional Riemann-Liouville derivative of order 1/2
Greek symbols		$D_t^{-1/2}$	– time-fractional Riemann-Liouville integral of order 1/2
α	– exponential pre-factor in eq. (14a), [s ⁻¹]	$x \rightarrow +0$	– x approaches zero from the right (right side medium)
β	– exponential pre-factor in eq. (14b), [s ⁻¹]	$x \rightarrow -0$	– x approaches zero from the left (left side medium)
δ_1, δ_2	– penetration depths, [m]		

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