ON HEAT CONDUCTION IN PERIODICALLY STRATIFIED COMPOSITES WITH SLANT LAYERING TO BOUNDARIES

by

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The paper deals with the modelling of heat conduction in periodically stratified composites with slant layering to boundaries. The relations of the homogenized model with microlocal parameters for the case of layering parallel to the boundary are transformed to conform to requirements of the composite geometry. The plane problem of periodically stratified half-space with slant layering heated by given boundary temperature is solved analytically. The obtained results are analysed and compared with the solutions for special cases: (1) homogenized half-space, (2) half-space with parallel layering to the boundary, and (3) half-space with vertical layering to the boundary.

Keywords: *temperature, heat flux, homogenized model, laminated composite*

Introduction

Determination of the temperature, heat flux distributions in non-homogeneous solids plays an important role in many engineering branches. A great deal of progress has been made to the studies of heat conduction problems in structures composed of periodically bounding together two or more materials with different thermal properties. Special attention has been devoted to laminated composites which represent an important type of modern materials. Moreover, the periodically stratified structures can be observed in many rocks and soils. The problems of modeling of heat conduction in laminated bodies were presented in numerous monographs and papers, see for example [1-11]. One of the approximated approaches to the description of heat conduction problems in periodically layered composites is the homogenized model with microlocal parameters [7-10]. The main merits of this model are that it permits to evaluate not only mean but also local values of temperature and heat fluxes in every component of the laminated body, as well as the thermal continuity conditions on interfaces are satisfied. The homogenized model with microlocal parameters has been applied in many boundary value problems for periodically stratified composites with layering parallel or perpendicular to the boundaries, see for instance [12-17]. The comparisons of the results obtained within the framework of the homogenized model with the solutions based on the classical heat conduction were presented in [18-20]. It can be observed good consistencies of results for both approaches in considered cases.

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In the present paper the homogenized model with microlocal parameters will be applied to heat conduction problems for the periodically two-layered composites with the boundaries inclined with an arbitrary angle to the layering. We confirm our considerations to the 2-D stationary heat conduction problems for the periodically layered half-space heated by given temperature on the boundary plane. The layering is assumed to be inclined to the boundary with an arbitrary angle. The continuity conditions of temperature and heat flux component normal to the layering, and the regularity conditions at infinity are taken into account. The problem will be approximated by using the homogenized model with microlocal parameters, which satisfies the continuity conditions on interfaces. The governing relations of this model were given in the terms of variables being the co-ordinates connected with the layering [7, 10, 12, 20]. However, it seems to be more useful to apply Cartesian co-ordinate system with axises normal and parallel to the boundary for solving the considered problems. In this paper the governing relations of the homogenized model with microlocal parameters [7, 10, 12-20] will be shortly recalled and next main of them will be presented on the co-ordinates connected with the boundary. By using Fourier transform methods the temperature distribution in the periodically stratified half-plane with inclined layering to the boundary heated by given boundary temperature will be found. The obtained results will be analysed by assumptions of some angles of the layering inclination to the boundary.

The determined problem is important in geophysics, engineering geology, because some soils or rocks have the periodically stratified structures (for instance varved clays, sandstone-slates, sandstone-shales). The governing relations of the homogenized model described in the terms of variables connected with the boundary can be developed for the 3-D non-stationary case.

Basic equations

Consider a non-homogeneous half-space composed of periodically repeated two-layered conductors with slant layering to the boundary plane. The continuity conditions of temperature and heat flux component normal to the layering are taken into considerations. Let (x, y, z) comprise the Cartesian co-ordinate system that y-axis is parallel to the layering, see fig. 1. Let K_1 and K_2 be the thermal conductivity of the subsequent component of the body, respectively. Moreover, let δ_1 and δ_2 denote the thicknesses of the layers being the constituents of composite, and $\delta = \delta_1 + \delta_2$ be the thickness of fundamental unit, fig.1.

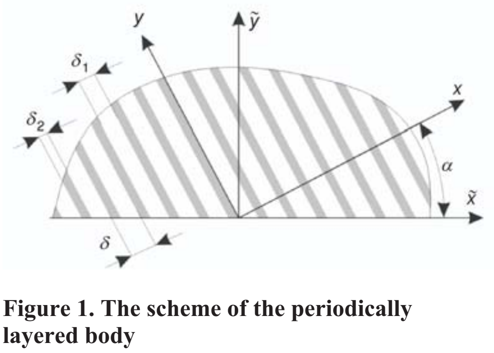

Figure 1. The scheme of the periodically layered body

Let the considered body be a half-space (or layer) with boundary plane inclined to the 0x axis with angle α, fig.1. Introduce the Cartesian co-ordinates $(\tilde{x},\tilde{y},\tilde{z})$ such that the axis $0\tilde{x}$ is inclined with angle α to the axis $0x$ and is located on the boundary plane. Let the periodically half-space $\tilde{y} > 0$ will be heated on the boundary $\tilde{y} = 0$ by given temperature $\vartheta(\tilde{x})$, $\tilde{x} \in R$, and the regularity conditions at infinity will be satisfied. Because of the boundary conditions on interfaces the exact solution of the heat conduction problem cannot be obtained, the homogenized model with microlocal parameters [7, 10] will be employed in order to seek an approximate solution. The relations of this model determined in the terms of (x, y) co-ordinates were presented in many papers, see for instance [7, 10, 12-20], so we recall only a brief outline of governing equations. The temperature and its gradient will be approximated as:

$$T(x, y) = \theta(x, y) + h(x)q(x, y) \approx \theta(x, y), \quad \frac{\partial T(x, y)}{\partial x} \approx \frac{\partial \theta(x, y)}{\partial x} + h'(x)q(x, y), \quad \frac{\partial T(x, y)}{\partial y} \approx \frac{\partial \theta(x, y)}{\partial y} \tag{1}$$

where $\theta(x, y)$ is an unknown macro-temperature, $q(x, y)$ – an unknown thermal micro-parameter, and $h(x)$ – the so-called shape function and it is given in the form:

$$h(x) = \begin{cases} x - 0.5\delta_1 & \text{for} \quad 0 \le x \le \delta_1 \\ \dfrac{-\eta x}{1-\eta} - 0.5\delta_1 + \dfrac{\delta_1}{1-\eta} & \text{for} \quad \delta_1 \le x \le \delta, \\ & h(x+\delta) = h(x) \end{cases} \tag{2}$$

where

$$\eta = \frac{\delta_1}{\delta} \tag{3}$$

Since for every x we have that $|h(x)| < \delta$, then for small δ the terms included the δ – periodic shape function are be small to and are neglected (the derivative $h'(x)$ is not small and the term included $h'(x)$ cannot be neglected). Moreover, the shape function (2) is chosen on such way, that the continuity conditions of heat flux component normal to the layering on interface, are satisfied.

The governing equations of the homogenized model for stationary 2-D case can be written as [18, 19]:

$$\tilde{K}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + [K]\frac{\partial q}{\partial x} = 0, \quad \tilde{K}q = -[K]\frac{\partial \theta}{\partial x} \tag{4}$$

where

$$\tilde{K} = \eta K_1 + (1-\eta)K_2, \quad [K] = \eta(K_1 - K_2), \quad \tilde{K} = \eta K_1 + \frac{\eta^2}{1-\eta}K_2 \tag{5}$$

The microlocal parameter q can be determined from $(4)_2$. It leads to:

$$\tilde{K}^{-1}K^* \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{6}$$

with

$$K^* = \frac{K_1 K_2}{(1-\eta)K_1 + \eta K_2} \tag{7}$$

The heat flux vector in a layer of the i-th, $i = 1, 2$, and expressed in the co-ordinate system (x, y) is given by:

$$q^{(i)}(x, y) \equiv (q_x^{(i)}, q_y^{(i)}, 0) = \left(-K^* \frac{\partial \theta}{\partial x}, \quad -K_i \frac{\partial \vartheta}{\partial y}, 0\right) \tag{8}$$

The considered boundaries are assumed to be inclined to the $0x$ axis with angle α. For this reason it will be more suitable to use co-ordinates $(\tilde{x}, \tilde{y})$, (see fig. 1). The relations between (x, y) and $(\tilde{x}, \tilde{y})$ can be written as:

$$x = \tilde{x}\cos\alpha + \tilde{y}\sin\alpha, \quad y = -\tilde{x}\sin\alpha + \tilde{y}\cos\alpha \tag{9}$$

After some calculations, eq.(6) for unknown macro-temperature $q(x, y)$ is transformed by using eq. (9) to the form:

$$\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{x}^2}(\tilde{K}^{-1}K^*\cos^2\alpha+\sin^2\alpha)+\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{x}\partial \tilde{y}}(\tilde{K}^{-1}K^*-1)\sin 2\alpha+$$
$$+\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{y}^2}(\tilde{K}^{-1}K^*\sin^2\alpha+\cos^2\alpha)=0 \tag{10}$$

The heat fluxes in the directions x and y given by (8) can be rewritten on the basis of eq. (9) in the form:

$$\begin{aligned} q_x^{(j)} &= -K^*\left(\cos\alpha\frac{\partial\theta(\tilde{x},\tilde{y})}{\partial\tilde{x}}+\sin\alpha\frac{\partial\tilde{\theta}(\tilde{x},\tilde{y})}{\partial\tilde{y}}\right), \\ q_y^{(j)} &= -K_i\left(-\sin\alpha\frac{\partial^2\theta(\tilde{x},\tilde{y})}{\partial\tilde{x}}+\cos\alpha\frac{\partial\tilde{\theta}(\tilde{x},\tilde{y})}{\partial\tilde{y}}\right)\end{aligned} \tag{11}$$

From eqs. (10) and (11) it can be observed that for the following cases we have:

– *Case 1*

Assume that:

$$K_1 = K_2 = K \tag{12}$$

then from eqs. (5) and (7) it follows that:

$$\tilde{K}=K, \quad [K]=0, \quad K^*=K \tag{13}$$

Equations (10) and (11) reduce to the form:

$$\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{x}^2}+\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{y}^2}=0 \tag{14}$$

and

$$\begin{aligned} q_x^{(1)} = q_x^{(2)} &= -K\left(\cos\alpha\frac{\partial\theta(\tilde{x},\tilde{y})}{\partial\tilde{x}}+\sin\alpha\frac{\partial\tilde{\theta}}{\partial\tilde{y}}\right), \\ q_y^{(1)} = q_y^{(2)} &= -K_i\left(-\sin\alpha\frac{\partial\theta(\tilde{x},\tilde{y})}{\partial\tilde{x}}+\cos\alpha\frac{\partial\tilde{\theta}}{\partial\tilde{y}}\right)\end{aligned} \tag{15}$$

which is agree with the well-known relations for the classical heat conduction problem in a homogeneous body.

– *Case 2*

Taking into account the angle:

$$\alpha = 0 \tag{16}$$

from eq. (10) and it follows that:

$$\tilde{K}^{-1}K^*\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{x}^2}+\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{y}^2}=0 \tag{17}$$

and

$$q_x^{(i)} = -K^*\frac{\partial\theta(\tilde{x},\tilde{y})}{\partial\tilde{x}^2}, \quad q_y^{(i)} = -K_i\frac{\partial\theta(\tilde{x},\tilde{y})}{\partial\tilde{y}}, \quad i=1,\ 2 \tag{18}$$

The obtained equations are agree with the relations of heat conduction in periodically two-layered composites with the boundary normal to the layering, see for instance [20, 21]. The $0x$ axis will be identical with $0\tilde{x}$ (and $0y$ axis will be agree with $0\tilde{y}$).

– *Case 3*

Assuming that:

$$\alpha = \frac{\pi}{2} \tag{19}$$

from eqs. (10) and (11) we obtain:

$$\frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{x}^2} + \tilde{K}^{-1}K^* \frac{\partial^2 \theta(\tilde{x},\tilde{y})}{\partial \tilde{y}^2} = 0 \tag{20}$$

and

$$q_x^{(i)} = -K^* \frac{\partial \theta(\tilde{x},\tilde{y})}{\partial \tilde{y}}, \quad q_y^{(i)} = -K_i \frac{\partial \tilde{\theta}(\tilde{x},\tilde{y})}{\partial \tilde{x}} \tag{21}$$

In this case we have $x = \tilde{y}$, $\tilde{x} = y$, and the obtained results (20) and (21) are agree with the relations of the homogenized model for periodically layered composites with the layering parallel to the boundary, see for example [14-18].

Formulation and solution of the considered problem

Consider the 2-D heat conduction problem for the periodically layered half-space $\tilde{y} > 0$ with the layering inclined with the angle $\pi/2 + \alpha$ to the boundary plane, fig. 1. The problem is assumed to be independent on the variable $\tilde{z}$, and stationary, and will be formulated within the framework of the homogenized model presented in the section *Basic equations*, so the continuity conditions of temperature and heat flux component normal to the layering are satisfied. The boundary $\tilde{y} = 0$ is heated by the given temperature $\vartheta(\tilde{x})$, $\tilde{x} \in R$, so the following boundary condition will be considered:

$$\theta(\tilde{x},0) = \vartheta(\tilde{x}), \quad \tilde{x} \in R \tag{22}$$

where satisfies:

$$\lim_{\tilde{x}\to\pm\infty} \vartheta(\tilde{x}) = 0 \tag{23}$$

Moreover, it will be assumed that:

$$\lim_{\tilde{x}\to\pm\infty} \tilde{\theta}(\tilde{x},\tilde{y}) = \lim_{\tilde{y}\to\infty} \frac{\partial \tilde{\theta}(\tilde{x},\tilde{y})}{\partial \tilde{x}} = 0 \tag{24}$$

The considered problem is described by eq. (10) and conditions (22) and (23). Using the integral Fourier transform with respect of variable:

$$\bar{\theta}(\gamma,\tilde{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \theta(\tilde{x},\tilde{y}) e^{-i\tilde{x}\gamma} d\tilde{x}, \quad i = \sqrt{-1} \tag{25}$$

from (10) it follows that $\bar{\theta}$ should satisfy the following ordinary differential equation:

$$(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha)\frac{d^2\bar{\theta}}{d\tilde{y}^2} + i\gamma \sin 2\alpha (\tilde{K}^{-1}K^* - 1)\frac{d\bar{\theta}}{d\tilde{y}} - \gamma^2 (\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha)\bar{\theta} = 0 \tag{26}$$

The general solution of eq. (26) takes the form:

$$\bar{\theta}(\gamma,\tilde{y}) = [D_1 \exp(-\omega\gamma\tilde{y}) + D_2 \exp(\omega\gamma\tilde{y})]\left\{\cos\left[\frac{\gamma \sin 2\alpha (\tilde{K}^{-1}K^* - 1)\tilde{y}}{2(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha)}\right] - \right.$$
$$\left. - i \sin\left[\frac{\gamma \sin 2\alpha (\tilde{K}^{-1}K^* - 1)\tilde{y}}{2(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha}\right]\right\} \tag{27}$$

where D_1 and D_2 are constants and:

$$\omega = \frac{\sqrt{\tilde{K}^{-1}K^*}}{\tilde{K}^{-1} \sin^2\alpha + \cos^2\alpha} \tag{28}$$

Using the conditions (22) and (24) and assuming that:

$$\vartheta(\tilde{x}) = \vartheta(-\tilde{x}), \quad x \in R \tag{29}$$

from eq. (27) it follows that the macro-temperature $\theta(\tilde{x}, \tilde{y})$ can be written in the form:

$$\theta(\tilde{x}, \tilde{y}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{\vartheta}(\gamma) \exp\left[-\frac{\sqrt{\tilde{K}^{-1}K^*}\gamma\tilde{y}}{\tilde{K}^{-1}K^* \sin^2 + \cos^2\alpha} \right] \cdot$$

$$\cdot \cos\left[\gamma \left(\tilde{y} \frac{\sin 2\alpha(\tilde{K}^{-1}K^* - 1)}{2(\tilde{K}^{-1}K^* \sin^2 + \cos^2\alpha)} - \tilde{x} \right) \right] \mathrm{d}\gamma \tag{30}$$

where

$$\bar{\vartheta}(\gamma) = \sqrt{\frac{2}{\pi}} \int_0^\infty \vartheta(\tilde{x}) \cos(\gamma\tilde{x}) \mathrm{d}\tilde{x} \tag{31}$$

The double integral in eq. (30) can be written in the form of convolution.

– *Special case*

Let the boundary temperature be taken into account as:

$$\vartheta(\tilde{x}) = \vartheta_0 H(|\tilde{x}| - a) \tag{32}$$

where ϑ_0 is a given constant and H is the Heaviside`s step function. From eq. (32) we obtain:

$$\bar{\vartheta}(\gamma) = \sqrt{\frac{2}{\pi}} \vartheta_0 \frac{\sin(a\gamma)}{\gamma} \tag{33}$$

Substituting (33) into (30) we have:

$$\theta(\tilde{x}, \tilde{y}) = \frac{2}{\pi} \vartheta_0 \int_0^\infty \frac{1}{\gamma} \exp\left[-\frac{\sqrt{\tilde{K}^{-1}K^*}\gamma\tilde{y}}{\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha} \right] \cdot$$

$$\cdot \cos\left[\gamma \left(\frac{\tilde{y} \sin 2\alpha \tilde{K}^{-1}K^* - 1}{2(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha} - \tilde{x} \right) \right] \sin(a\gamma) \mathrm{d}\gamma \tag{34}$$

This integral can be calculated analytically. Because [21]

$$\int_0^\infty \frac{1}{x} \mathrm{e}^{-px} \cos(bx) \sin(ax) \mathrm{d}x = \frac{1}{2} \tan^{-1}\left(\frac{2pa}{p^2 - a^2 + b^2} \right) + \frac{\pi}{2} s \tag{35}$$

where

$$s = \begin{cases} 0 & \text{for} \quad p^2 - a^2 + b^2 > 0 \\ 1 & \text{for} \quad p^2 - a^2 + b^2 < 0 \end{cases} \tag{36}$$

thus we obtain:

$$\theta(\tilde{x}, \tilde{y}) = \frac{1}{\pi} \vartheta_0 \tan^{-1} \left\{ 2a\tilde{y} \frac{\sqrt{\tilde{K}^{-1}K^*}}{\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha} \left[\frac{\tilde{K}^{-1}K^* \tilde{y}^2}{(\tilde{K}^{-1}K^* \sin^2 + \cos^2\alpha)^2} - a^2 + \right. \right.$$

$$\left. \left. + \left(\frac{\tilde{y} \sin 2\alpha(\tilde{K}^{-1}K^* - 1)}{2(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha} - \tilde{x} \right)^2 \right]^{-1} \right\} + \vartheta_0 s \tag{37}$$

where

$$s = 0 \ \text{ for } \ \frac{\tilde{K}^{-1}K^* \tilde{y}^2}{(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha)} - a^2\alpha \left(\frac{\tilde{y} \sin 2\alpha(\tilde{K}^{-1}K^* - 1)}{2(\tilde{K}^{-1}K^* \sin^2\alpha + \cos^2\alpha)} - \tilde{x} \right)^2 > 0 \tag{38}$$

and $s = 1$ if the opposite inequality to the above ones is valid.

– *Case 1*

It can be observed that for the homogeneous body described by (12) and (13) from (37) and (38) it follows that

$$\theta(\tilde{x},\tilde{y}) = \frac{1}{\pi}\vartheta_0 \tan^{-1}\frac{2a\tilde{y}}{\tilde{y}^2 - a^2 + \tilde{x}^2} + s\vartheta_0 \tag{39}$$

where

$$s = 0 \text{ for } \tilde{x}^2 + \tilde{y}^2 > a^2, \text{ and } s = 1 \text{ for } \tilde{x}^2 + \tilde{y}^2 < a^2 \tag{40}$$

Equation (39) together with (40) are agree with the adequate result obtained directly for the homogeneous half-plane.

– *Case 2*

If the layering is normal to the boundary, what it leads to $\alpha = 0$ from eq. (37) we obtain:

$$\theta(\tilde{x},\tilde{y}) = \frac{1}{\pi}\vartheta_0 \tan^{-1}\left(\frac{2a\tilde{y}\sqrt{\tilde{K}^{-1}K^*}}{\tilde{K}^{-1}K^*\tilde{y}^2 - a^2 + \tilde{x}^2}\right) + s\vartheta_0 \tag{41}$$

where

$$s = 0 \text{ for } \tilde{x}^2 + \tilde{K}^{-1}K^*\tilde{y}^2 > a^2, \text{ and } s = 1 \text{ for } \tilde{x}^2 + \tilde{K}^{-1}K^*\tilde{y}^2 < a^2 \tag{42}$$

– *Case 3*

In the case of $\alpha = \pi/2$ (the layering is parallel to the boundary) from eq. (37) it follows that:

$$\theta(\tilde{x},\tilde{y}) = \frac{1}{\pi}\vartheta_0 \tan^{-1}\left(\frac{2a\tilde{y}\sqrt{\tilde{K}^{-1}K^*}}{\tilde{y}^2 + \tilde{K}^{-1}K^*(\tilde{x}^2 - a^2)}\right) + s\vartheta_0 \tag{43}$$

where

$$s = 0 \text{ for } \tilde{y}^2 + \tilde{K}^{-1}K^*(\tilde{x}^2 - a^2) > 0, \text{ and } s = 1 \text{ for } \tilde{y}^2 + \tilde{K}^{-1}K^*(\tilde{x}^2 - a^2) < 0 \tag{44}$$

The knowledge of temperature distributions in the periodically layered half-space described in the co-ordinates $(\tilde{x},\tilde{y})$ permits to calculate the macro-temperature θ in the co-ordinates (x, y) connected with the layering by using inverse transformation to the relation (9). We have:

$$\tilde{x} = x\cos\alpha - y\sin\alpha, \quad \tilde{y} = x\sin\alpha + y\cos\alpha \tag{45}$$

The co-ordinates (x, y) seem to be more useful to determine the heat flux vector in every lamina on the basis of equations (8), and (30), or (37) and (45).

Numerical results

The temperature distributions given in eqs. (37) and (38) can be presented graphically. Introducing the dimensionless co-ordinates $(\tilde{x}^*, \tilde{y}^*)$ as:

$$\tilde{x}^* = \frac{\tilde{x}}{a}, \quad \tilde{y}^* = \frac{\tilde{y}}{a}, \quad \text{and} \quad \tilde{\delta} = \frac{\delta}{a} \tag{46}$$

from eqs.(37) and (38) it follows that:

$$\theta(\tilde{x}^*,\tilde{y}^*) = \frac{1}{\pi}\vartheta_0 \tan^{-1}\left\{2a\tilde{y}^*\frac{\sqrt{\tilde{K}^{-1}K^*}}{\tilde{K}^{-1}K^*\sin^2\alpha + \cos^2\alpha}\left[\frac{\tilde{K}^{-1}K^*\tilde{y}^{*2}}{(\tilde{K}^{-1}K^*\sin^2\alpha + \cos^2\alpha)^2} - 1 + \right.\right.$$
$$\left.\left. + \left(\frac{\tilde{y}^*\sin 2\alpha(\tilde{K}^{-1}K^* - 1)}{2(\tilde{K}^{-1}K^*\sin^2\alpha + \cos^2\alpha} - x^*\right)^2\right]^{-1}\right\} + s \tag{47}$$

where

$$s=0 \ \text{ for } \ \frac{\widetilde{K}^{-1}K^{*}\widetilde{y}^{*2}}{(\widetilde{K}^{-1}K^{*}\sin^{2}\alpha+\cos^{2}\alpha)^{2}}-1+\left[\frac{\widetilde{y}^{*}\sin 2\alpha(\widetilde{K}^{-1}K^{*}-1)}{2(\widetilde{K}^{-1}K^{*}\sin^{2}\alpha+\cos^{2}\alpha)}-x^{*}\right]^{2}>0 \quad (48)$$

and $s = 1$ for the opposite inequality to the above ones.

The isothermal lines for the homogeneous half-plane ($K_1/K_2 = 1$) and the periodically layered half-plane with angles $\alpha = 0$, $\pi/4$, $\pi/2$, and $K_1/K_2 = 4$, are shown in fig. 2.

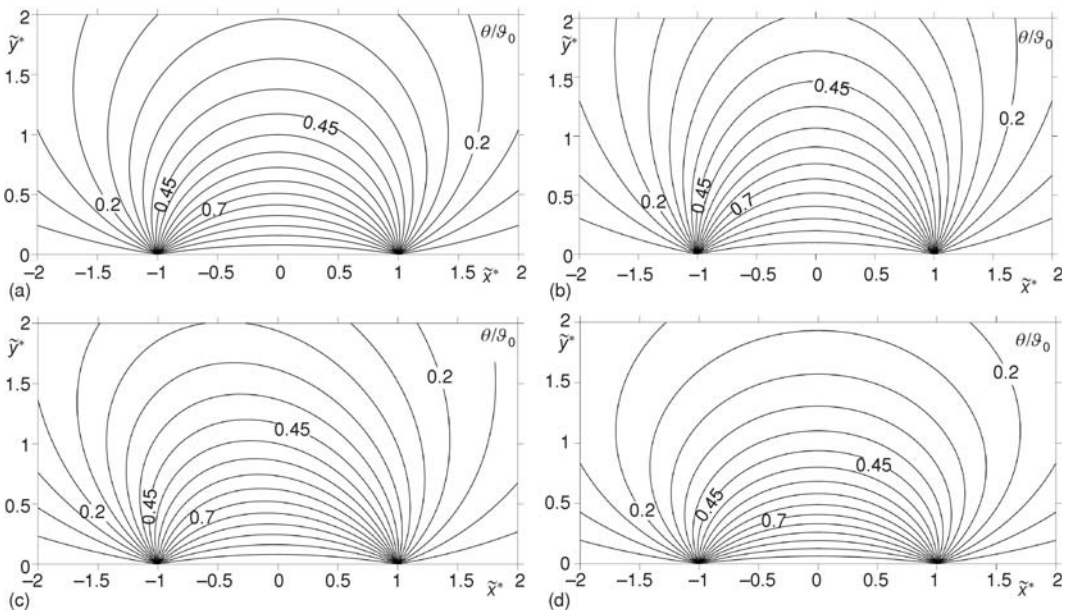

Figure 2. The isothermal lines for the considered half-plane for: (a) $K_1/K_2 = 1$, (b) $\alpha = 0$, (c) $\alpha = \pi/4$, (d) $\alpha = \pi/2$, and $K_1/K_2 = 4$, $\eta = 0.5$

The dimensionless temperature θ/ϑ_0 as a function of angle α at the point $\widetilde{x}^* = 0$, $\widetilde{y}^* = 0.05$ (under the centre of heated boundary region) for $K_1/K_2 = 1$, 4, and 8, and $\eta = 0.5$ is presented in fig. 3(a). It is seen that the highest values of temperature at this point are achieved for $\alpha = 0$ (the layering is perpendicular to the boundary) and the smallest values for $\alpha = \pi/2$ (the layering is parallel to the boundary). Figure 3(b) shows the dimensionless temperature θ/ϑ_0 as a function of parameter η for $\alpha = 0$, π, and $\pi/2$, and the $K_1/K_2 = 4$ and 8, at the same point as in fig. 3(a). The maximal or minimal values of the dimensionless temperature are achieved for $\eta = 0.5$.

The dimensionless components of heat flux vector are presented in figs. 4-6. The component $q_x a/(\vartheta_0 K^*)$ as a function of $\widetilde{x}^*$ (see fig. 1) on the depths $\widetilde{y}^* = 0.05$, 0.25, 050 for $K_1/K_2 = 1$ (the homogeneous half-plane) as well as $K_1/K_2 = 4$, $\eta = 0.5$ and three cases of angle α, $\alpha = 0$, $\pi/4$, and $\pi/2$ is shown in fig. 4.

The components $q_x^{(i)}$, $i = 1, 2$ are continuous on the interfaces, so we denote $q_x = q_x^{(i)}$. It can be observed that the extreme values of q_x are achieved at points $|\widetilde{x}^*| = 1$ (under the ends of heated range). For the angle of layering inclination $\alpha = 0$ (the layering is normal to the boundary) q_x is positive for $\widetilde{x}^* > 0$, but when $\alpha = \pi/4$ or $\alpha = \pi/2$ the component q_x changes sign.

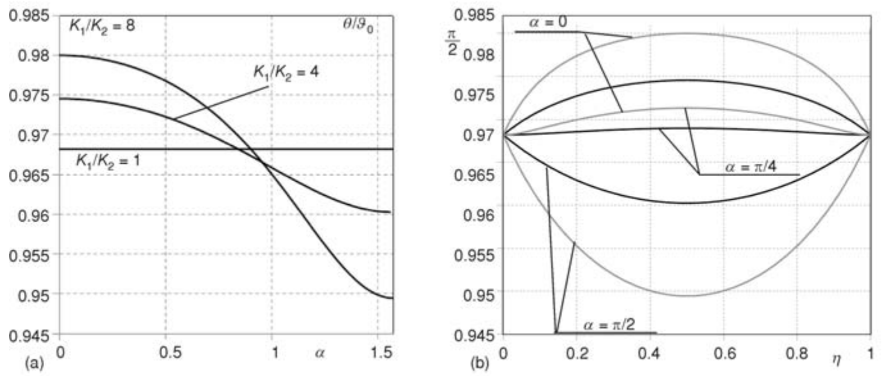

Figure 3. The dimensionless temperature q/ϑ_0 at the point $\tilde{x}^* = 0$ $\tilde{y}^*$=0.05: (a) as a function of α, (b) as a function of η, where black line is for K_1/K_2 = 4 and grey line is for K_1/K_2 = 8

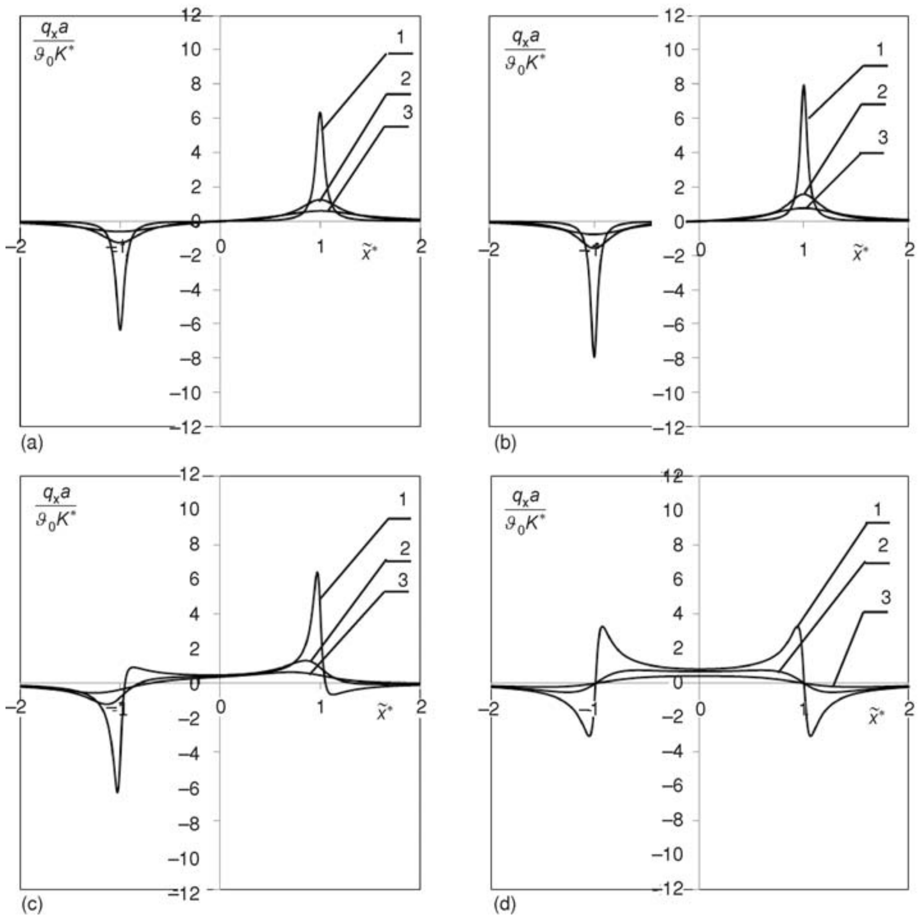

Figure 4. The dimensionless component of heat flux $q_x^{(i)}a/(\vartheta_0 K^*)$, i = 1, 2, as a function of $\tilde{x}^*$: (a) K_1/K_2 = 1, (b) α = 0, (c) α = π/4, (d) α = π/2, and K_1/K_2 = 4, η = 0.5, 1– $\tilde{y}^*$= 0.05, 2 – $\tilde{y}^*$= 0.25, 3 – $\tilde{y}^*$= 0.50

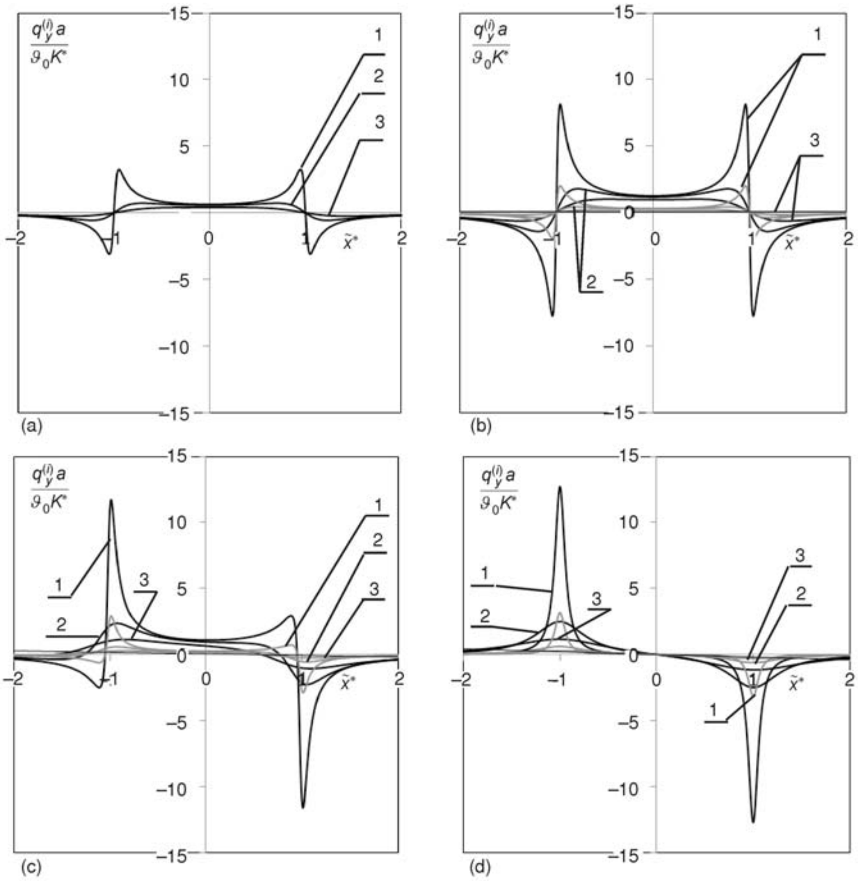

Figure 5. The dimensionless component of heat flux $(q_y^{(i)}a)/(\vartheta_0 K^*)$, i = 1, 2, as a function of $\tilde{x}^*$: (a) $K_1/K_2 = 1$, (b) $\alpha = 0$, (c) $\alpha = \pi/4$, (d) $\alpha = \pi/2$, and $K_1/K_2 = 4$, $\eta = 0.5$, 1 – $\tilde{y}^* = 0.05$, 2 – $\tilde{y}^* = 0.25$, 3 – $\tilde{y}^* = 0.50$, and i = 1 – black line; i = 2 – grey line

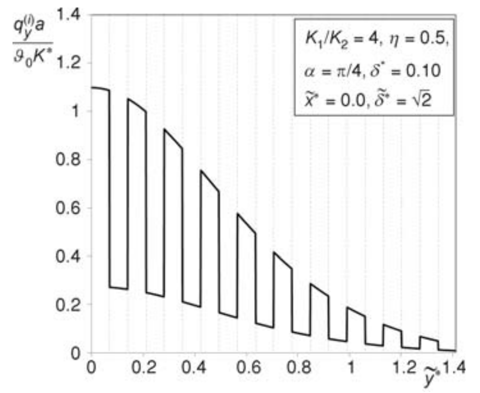

Figure 6. The dimensionless component of heat flux $(q_y^{(i)}a)/(\vartheta_0 K^*)$, i = 1, 2, as a function of $\tilde{y}^*$ for $\tilde{x}^* = 0.0$

The components $q_y^{(i)}$, $i = 1, 2$ are discontinuous on the interfaces. Figure 5 shows the dimensionless component of flux vector $(q_y^{(i)}a)/(\vartheta_0 K^*)$ as a function of $\tilde{x}^*$ on the depths $\tilde{y}^* =$ 0.05, 0.25, and 0.50 for homogeneous body $K_1/K_2 = 1$ (in this case $q_y^{(1)} = q_y^{(2)}$ as well as for $K_1/K_2 = 4$, $\eta = 0.5$, and $\alpha = 0$, $\pi/4$, and $\pi/2$. The black curves represent the heat flux $q_y^{(1)}$ (in the layers of first kind), the grey curves correspond with $q_y^{(1)}$, and both kinds of curves are extended for a better visibility.

To show the jumps of the heat flux $q_y^{(i)}$ fig. 6 is added. Figure 6 presents the dimensionless $(q_y^{(i)}a)/(\vartheta_0 K^*)$ as a function of $\tilde{y}^*$ (the depth) for $\tilde{x}^* = 0.0$, $K_1/K_2 = 4$, $\eta = 0.5$, $\alpha = \pi/4$, $\delta = 1$, then

$\tilde{\delta}^* = 2^{1/2}\delta/a$. It can be observed that $q_y^{(1)} \neq q_y^{(2)}$ for $\tilde{y} = n\tilde{\delta}^*$ (on the interfaces), and the jumps are greatest near the boundary plane.

Final remarks

The presented results for temperature and heat distributions in the periodically layered half-plane with the layering inclined to the boundary possess the same characteristics as adequate solutions within the framework of the classical description. The temperature and heat flux are continuous and component $q_y^{(i)}$ is discontinuous on the interfaces, By using co-ordinates $(\tilde{x}, \tilde{y})$ connected with boundary, then both components of heat fluxes in the $\tilde{x}$, and $\tilde{y}$ -directions experience jumps on the interfaces.

The derived relations given by (10) and (11) permit to solve by analytical methods some boundary value problems for heat conduction in laminated composites with slant layering to boundaries.

Nomenclature

a – half of length of heated range, [m]
$h(x)$ – shape function of the homogenized model with microlocal parameters, [m]
i – complex unit $(= -1^{1/2})$, [–]
K_1, K_2 – coefficients of thermal conductivity of the subsequent component of the body, [Wm^{-1}K^{-1}]
$\tilde{K}, K^*$ – effective thermal modulus on the homogenized model with microlocal parameters, [Wm^{-1}K^{-1}]
q – unknown thermal micro-parameter of the homogenized model with microlocal parameters, [Km^{-1}]
$q^{(i)}$ – heat flux vector in a layer of the i-th kind, $i = 1, 2$, [W]
x, y – Cartesian co-ordinates connected with the layering, [m]
$\tilde{x}, \tilde{y}$ – Cartesian co-ordinates connected with the boundary, [m]
$(\tilde{x}^*, \tilde{y}^*)$ – Cartesian dimensionless co-ordinates, [–]
T – temperature, [K]

Greek symbols

α – angle of inclination of layering to axis [rad],
δ_1, δ_2 – thickness of the layers being the constituents of composite, [m]
δ – thickness of fundamental unit, $(= \delta_1 + \delta_2)$, [m]
η – saturation coefficient of fundamental unit by the first kind of material, $(= \delta_1/\delta)$, [–]
θ – macro-temperature in homogenized model with microlocal parameters, [K]
$\vartheta(\tilde{x})$ – boundary temperature, [K]

Indexes

i – kind of sublayer; $i = 1$ the first kind or $i = 2$ the second kind of the subsequent layers

References

[1] Aurialt, J. L., Effective Macroscopic Description for Heat Conduction in Periodic Composites, *International Journal of Heat and Mass Transfer, 26* (1983), 6, pp. 861-869
[2] Backhalov, N. S., Panasenko, G. P., *Averaged Processes in Periodic Media* (in Russian), Nauka, Moscow, 1984
[3] Bedford, A., Stern, M., Toward Diffusing Continuum Theory of Composite Materials, *Journal of Applied Mechanics, 38* (1971), 1, pp. 8-14
[4] Bufler, H., Stationary Heat Conduction in a Macro- and Microperiodically Layered Solid, *Archives of Applied Mechanics, 70* (2000), 1-3, pp. 103-114
[5] Furmasnki, P., Heat Conduction in Composites: Homogenization and Macroscopic Behavior, *Applied Mechanics Reviews, 50* (1997), 6, pp. 327-356
[6] Ignaczak, J., Baczynski, Z. F., On a Refined Heat-Conduction Theory of Microperiodic Layered Solids, *Journal of Thermal Stresses, 20* (1997), 7, pp. 749-771
[7] Wozniak, Cz., A Nonstandard Method of Modeling of Thermoelastic Composites, *International Journal of Engineering Science, 25* (1987), 5, pp. 483-499

[8] Maewal, A., Homogenization for Transient Heat Conduction, *ASME Journal of Applied Mechanics, 46* (1979), 4, pp. 945-946
[9] Manevith, L. I., *et al*., *Mechanics of Periodically Layered Heterogenous Structures*, Springer, Berlin, 2002
[10] Matysiak, S. J., Wozniak, Cz., On the Modelling of Heat Conduction Problem in Laminated Bodies, *Acta Mechanica, 65* (1986), 1-4, pp. 223-238
[11] Wozniak, Cz., Wierzbicki, E., *Averaging Techniques in Thermomechanics of Composite Solids. Tolerance Averaging versus Homogenization,* Wydawnictwo Politechniki Czestochowskiej, Czestochowa, Poland, 2000
[12] Matysiak, S. J., *et al*., Temperature Field in a Microperiodic Two -layered Composite Caused by a Circular Laser Heat Source, *Heat and Mass Transfer, 34* (1998), 2-3, pp. 127-133
[13] Matysiak, S. J., *et al*., Distribution of Friction Heat during Cold – Rolling of Metals by Using Composite Rolls, *Numerical Heat Transfer, Part A, 34* (1998), 3, pp. 719-729
[14] Matysiak, S. J., Pauk, V., Plane Contact Problem for Periodic Laminated Composite Involving Frictional Heating, *Archive of Applied Mechanics, 66* (1995), 1-2, pp. 82-89
[15] Kaczynski, A., Matysiak,,S. J., On the Three Dimensional Problem of an Interface Crack under Uniform Heat Flow in a Bimaterial Periodically-Layered Space, *International Journal of Fracture, 123* (2003), 3-4, pp. 127-138
[16] Kaczynski, A., Matysiak, S. J., Thermal Stresses in a Laminate Composite with a Row of Interface Cracks, *International Journal of Engineering Science, 27* (1989), 2, pp. 131-147
[17] Matysiak, S. J., *et al*., Temperature Field in the Process of Braking of a Massive Body with Composite Coating, *Materials Sciences, 43* (2007), 1, pp. 62-69
[18] Kulchytsky-Zhyhailo, R., Matysiak, S. J., On Heat Conduction Problem in a Semi-Infinite Periodically Laminated Layer, *International Communications in Heat and Mass Transfer, 32* (2005), 1-2, pp. 123-132
[19] Kulchytsky-Zhyhailo, R., Matysiak, S. J., On Some Heat Conduction Problem in a Periodically Two-Layered Body: Comparative Results, *International Communications in Heat and Mass Transfer, 32* (2005), 3-4, pp. 332-340
[20] Matysiak, S. J., Perkowski, D. M., On Heat Conduction in a Semi-Infinite Laminated Layer. Comparative Results for Two Approaches, *International Communications in Heat and Mass Transfer, 37* (2010), 4, pp. 343-349
[21] Gradshteyn, I. S., Ryzhik, I. M., *Tables of Integrals, Series and Products* (in Russian), Science, Moscow, 1971

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