

NUMERICAL APPROACH TO STEFAN PROBLEM IN A TWO-REGION AND LIMITED SPACE

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In the paper, an effective numerical scheme for two-region and limited space Stefan problem is presented. The main feature of this method is to search for its solution and the temperature field simultaneously. The comparison of the obtained results with approximate analytical solutions shows that the algorithm is feasible and effective.

Key words: two-region and limited space Stefan problem, moving interface, numerical simulation

Introduction

As a non-linear heat transfer problem, the Stefan problem has received considerable attention by researchers for its important applications in many engineering fields [1-6]. Reference [7] presented an algorithm for the classic one-region Stefan problem. This paper suggests an efficient numerical algorithm for two-region case to deal with the moving interface.

Governing equations and numerical scheme

The solidification of a slab (two-region problem) will be used as an example to illustrate the algorithm. A liquid at a temperature T_l which is higher than the solidification temperature T_m is confined in a space of finite thickness, $0 \leq x \leq L$. For times $t > 0$ the boundary surface at $x = 0$ is maintained at a constant temperature T_0 below T_m and the boundary at $x = L$ is kept insulated. The solidification starts at the surface $x = 0$. The location of the solid-liquid interface is to be determined as a function of time. Figure 1 shows the geometry and the temperature profiles. The mathematical formulation of this problem for the solid phase is given as:

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{a_s} \frac{\partial T_s}{\partial t} \quad \text{in} \quad 0 < x < h(t), t > 0 \quad (1)$$

$$T_s(x, t) = T_0 \quad \text{at} \quad x = 0, t > 0 \quad (2)$$

For the liquid phase as:

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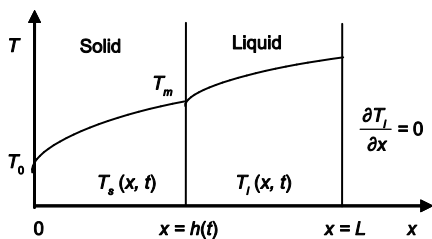


Figure 1. Solidification of a slab
(two-region problem)

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{a_l} \frac{\partial T_l}{\partial t} \quad \text{in} \quad h(t) < x < L, t > 0 \quad (3)$$

$$T_l(x, t) = T_i \quad \text{for} \quad t = 0, \text{ in } 0 < x < L \quad (4)$$

$$\frac{\partial T_l}{\partial x} = 0 \quad \text{at} \quad x = L, t > 0 \quad (5)$$

and the coupling conditions at the interface $x = s(t)$ are:

$$T_s(x, t) = T_l(x, t) = T_m \quad \text{at} \quad x = h(t), t > 0 \quad (6)$$

$$k_s \frac{\partial T_s}{\partial x} - k_l \frac{\partial T_l}{\partial x} = \rho r \frac{dh}{dt} \quad \text{at} \quad x = h(t), t > 0 \quad (7)$$

where T_s , T_l , k_s , k_l , a_s and a_l are the temperature, the thermal conductivity and the diffusivity of the solid and the liquid phase, respectively. L is the thickness of the slab.

No exact solution is available for the above discussed problem except for an approximate integral solution. We will establish the numerical method and then compare the numerical result with the approximate solution.

Numerical scheme

A slightly different from the method presented in [7], the moving interface positions at different time are preset and assigned. For the sake of simplicity, these positions are assumed to be equally spaced intervals. The mesh sketch is shown in fig. 2. These positions are denoted by h_2, h_3, \dots, h_{n+1} corresponding to arrival time t_1, t_2, \dots, t_n , and then these positions co-ordinates are used as the computing mesh grids to determine the arrival time of each respective moving interface. Let $x_1 = h_1 = 0$, $x_i = h_i$, $\Delta x = L/N = x_i - x_{i-1} = h_i - h_{i-1}$ ($i = 2, 3, 4, \dots, n$), $\Delta t_1 = t_1$, $\Delta t_2 = t_2 - t_1$, and $\Delta t_n = t_n - t_{n-1}$, where Δx is the space step, Δt_n – the n -th time step, and N – the number of the space interval.

Forward or backward difference formulation and central difference formulation are used for the first and second-order derivatives singly throughout the method. The numerical procedure is as follows.

Determination of the time step Δt_i

As showed in fig. 3 the star mark denotes the arrival position of the moving interface at the time step Δt_1 and divides the computational region into two parts, *i. e.* solid and liquid regions. The discrete form of eq. (7) is:

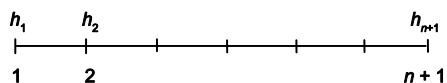


Figure 2. The mesh sketch for the
solidification of a slab

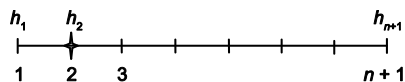


Figure 3. The mesh sketch to determine
time step Δt_1

$$k_s \frac{T_2^1 - T_1^1}{\Delta x} - k_l \frac{T_3^1 - T_2^1}{\Delta x} = \rho r \frac{\Delta x}{\Delta t_1}$$

Here the superscript and subscript denote the time step and grid numbers respectively. Let $\beta_k = k_l/k_s$, $\beta_a = a_l/a_s$ and $\Omega_s = \rho r \Delta x^2/k_s$, then the above expression can be rewritten as:

$$[(1 + \beta_k)T_m - T_0 - \beta_k T_3^1] = \frac{\Omega_s}{\Delta t_1}$$

The temperatures of node 1 and node 2 in the solid region are already known, *i. e.* $T_1^1 = T_0$ and $T_2^1 = T_m$, so only the node temperatures in the liquid region are to be determined.

The finite difference formulation of eq. (3) at node 3 is:

$$\frac{T_3^1 - T_3^0}{\Delta t_1} = a_l \frac{T_2^1 - 2T_3^1 + T_4^1}{\Delta x^2}$$

Substituting $T_2^1 = T_m$, $T_3^0 = T_i$ into above express yields:

$$\left(2 + \frac{R_l}{\Delta t_1}\right) T_3^1 - T_4^1 = \frac{R_l T_i}{\Delta t_1} + T_m$$

Generally, at nodes $k-n$ we have:

$$-T_{k-1}^1 + \left(2 + \frac{R_l}{\Delta t_1}\right) T_k^1 - T_{k+1}^1 = \frac{R_l T_i}{\Delta t_1}$$

At node $n+1$, according to the adiabatic boundary condition on $x = L$ we have:

$$T_n^1 = T_{n+1}^1$$

Finally, we have the following equations:

$$\begin{aligned} [(1 + \beta_k)T_m - T_0 - \beta_k T_3^1] &= \frac{\Omega_s}{\Delta t_1} \\ \left(2 + \frac{R_l}{\Delta t_1}\right) T_3^1 - T_4^1 &= \frac{R_l T_i}{\Delta t_1} + T_m \\ &\dots\dots\dots \\ -T_{k-1}^1 + \left(2 + \frac{R_l}{\Delta t_1}\right) T_k^1 - T_{k+1}^1 &= \frac{R_l T_i}{\Delta t_1} \\ &\dots\dots\dots \\ T_n^1 &= T_{n+1}^1 \end{aligned}$$

The unknowns $T_3^1, T_4^1, \dots, T_{n+1}^1$, and Δt_1 will be determined by the n equations, hence, arrival time t_1 ($\Delta t_1 = t_1$) and the node temperatures can be obtained simultaneously. The algorithm for solving the equation set is discussed later in this section.

Determination of the time step Δt_k

Using the similar steps, we have the following equations:

$$[(1 + \beta_k)T_m - T_k^k - \beta_k T_{k+2}^k] = \frac{\Omega_s}{\Delta t_k} \quad (8)$$

For the solid phase:

$$A_1 \quad T = B_1 \quad (9)$$

where

$$A_1 = \begin{bmatrix} \left(2 + \frac{R_s}{\Delta t_k}\right) & -1 & 0 & & & \\ & -1 & \left(2 + \frac{R_s}{\Delta t_k}\right) & -1 & 0 & \\ & \dots & \dots & \dots & \dots & \dots \\ & & -1 & \left(2 + \frac{R_s}{\Delta t_k}\right) & -1 & 0 \\ & & & -1 & \left(2 + \frac{R_s}{\Delta t_k}\right) & \end{bmatrix}, \quad T = \begin{bmatrix} T_2^k \\ T_3^k \\ \dots \\ T_{k-1}^k \\ T_k^k \end{bmatrix}, \quad B_1 = \begin{bmatrix} T_0 + \frac{R_s T_2^{k-1}}{\Delta t_k} \\ \frac{R_s T_3^{k-1}}{\Delta t_k} \\ \dots \\ \frac{R_s T_{k-1}^{k-1}}{\Delta t_k} \\ T_m + \frac{R_s T_k^{k-1}}{\Delta t_k} \end{bmatrix},$$

and for the liquid phase:

$$A_2 \quad T = B_2 \quad (10)$$

where

$$A_2 = \begin{bmatrix} \left(2 + \frac{R_l}{\Delta t_k}\right) & -1 & 0 & & & \\ & -1 & \left(2 + \frac{R_l}{\Delta t_k}\right) & -1 & 0 & \\ & \dots & \dots & \dots & \dots & \dots \\ & & -1 & \left(2 + \frac{R_l}{\Delta t_k}\right) & -1 & 0 \\ & & & -1 & \left(2 + \frac{R_l}{\Delta t_k}\right) & \end{bmatrix}, \quad T = \begin{bmatrix} T_{k+2}^k \\ T_{k+3}^k \\ \dots \\ T_n^k \\ T_{n+1}^k \end{bmatrix}, \quad B_2 = \begin{bmatrix} T_m + \frac{R_l T_{k+2}^{k-1}}{\Delta t_k} \\ \frac{R_l T_{k+3}^{k-1}}{\Delta t_k} \\ \dots \\ \frac{R_l T_n^{k-1}}{\Delta t_k} \\ 0 \end{bmatrix}$$

We note that the matrix of eqs. (9) and (10) is a tri-diagonal matrix characterized by predominantly main diagonal elements so they can be readily solved by the Thomas algorithm. The solution procedure involves:

- using a guessing time value Δt_k^* (for instance the time value calculated by the exact solution of a half-space Stefan problem) to calculate the matrix $[A_1]$, $[A_2]$, $[B_1]$ and $[B_2]$, respectively.

- solving eqs. (9) and (10) to obtain all the node temperatures.
- taking the eq. (8) as the convergence criterion:

$$\text{Ref} = \left| (1 + \beta_k)T_m - T_k^k - \beta_k T_{k+2}^k - \frac{\Omega_s}{\Delta t_k^*} \right| \leq \varepsilon \quad (11)$$

If eq. (11) is not satisfied, then a new time value is assumed again. Repeat the steps until convergence is reached.

Calculation example

The solidification of aluminum is considered. The thickness of the molten aluminum is taken as 0.1 meter. Let $N = 20$, thus $\Delta x = \Delta h = 0.005$ m. The other physical parameters are given as following [8]: $T_i = 1073$ K, $T_0 = 573$ K, $r = 396 \cdot 10^3$ J/kg, $\rho_l = 2380$ kg/m³, $k_l = 215$ W/mK, $c_l = 1130$ J/kgK, $\rho_s = 2545$ kg/m³, $k_s = 225.5$ W/mK, and $c_s = 1016$ J/kgK. The convergence criterion is set as $\varepsilon = 1 \cdot 10^{-3}$.

According to the integral approximate solution [9], the location of the moving interface as a function of time is:

$$h(t) = 2\lambda\sqrt{a_s t} \quad (12)$$

For this example, the value of λ is 0.4856 and valid computing length of time of the approximate solution is $t \leq 11.66$ (second).

The calculated results are shown in figs. 4 and 5. Figure 4 demonstrates the numerical results are coincident closely with the approximate solution which can be obtained by the homotopy perturbation method [10, 11] and the variational iteration method [12]. The temperature distribution as a function of time is shown in fig. 5. Owing to the high conductivity, the temperature in the solid region is almost linearly distributed.

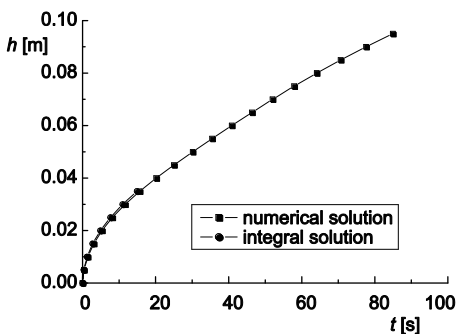


Figure 4. The comparison of numerical results with approximate integral solution

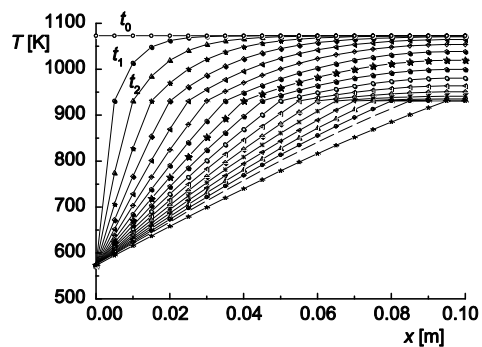


Figure 5. The temperature distribution vs. time

Conclusions

A numerical method for two-region and limited space Stefan problem is presented in the paper. The main focus in this method is to solve the time variable and the temperature field simultaneously by presetting and assigning the moving interface positions. Through this

approach, the difficulty in mesh generation for moving-boundary problem can be avoided completely.

Especially, the method aims at the solution of the time variable, so no time step length is involved to be considered and the computational efficiency will be improved greatly compared with the front tracking method in which the time step must be small enough to insure the numerical process convergence.

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