THERMAL IMPEDANCE ESTIMATIONS BY SEMI-DERIVATIVES AND SEMI-INTEGRALS
1-D semi-infinite cases

by

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Simple 1-D semi-infinite heat conduction problems enable to demonstrate the potential of the fractional calculus in determination of transient thermal impedances under various boundary conditions imposed at the interface (x = 0). The approach is purely analytic and very effective because it uses only simple semi-derivatives (half-time) and semi-integrals and avoids development of entire domain solutions.

Key words: *thermal impedance, fractional calculus, half-time fractional derivatives*

Introduction

A thermal source heating a half-space is a general academic formulation of high temperature source interaction with materials in the material processing and precision manufacturing. Although this is a very old subject [1, 2], it remains scientifically important and at the microscopic scale, difficult to master in some configurations [3], such as transitional arrangements and the presence of heat convection, cooling of electronic devices both at the package and system level, and cooling of power semi-conductors using heat sinks [4]. The main problem emerging under such circumstances is about the transient material behaviour and the heat spreading in depth of the processed materials [2, 5, 6]. In many cases, the time-depended behaviour relevant to the thermal impedance is of primary importance [7-10]. Some works on transient temperature field dues to heat spots with uniform [3] or continuous or time-dependent heat sources [11] have been developed. The thermal impedance has been estimated by either exact analytical [1, 3, 5, 6, 9, 12-14] or numerical solutions [4, 9].

Case 1

Concerning the 1-D energy equation in a heated semi-infinite materials heated by a line (plane) isothermal source at $x = 0$, we have:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad T = T(x, t) \tag{1}$$

with boundary and initial conditions:

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = \begin{cases} q_s, & x = 0, \quad t > 0, \\ 0, & x \geq 0, \quad t > 0, \end{cases} \qquad T(\infty, t) = T_\infty, \quad T(x,0) = T_\infty,\ t = 0 \tag{2}$$

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$T(x, 0)$ indicates an undisturbed initial temperature field of the heated materials.

The solution eq. (1) is well known [1]:

$$T(x,t) = T_\infty + \frac{2q_\mathrm{s}}{\lambda}\sqrt{\alpha t}\left\{\frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{x}{2\sqrt{\alpha t}}\left[1-\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]\right\} \tag{3}$$

Setting $x = 0$ we get the surface temperature $T_\mathrm{s} = T(0, t)$ that allows to define the thermal impedance of the semi-infinite medium $Z = (T_\mathrm{s} - T_\infty)/q_\mathrm{s}$. In this simple case we have $Z_1 = (2\sqrt{at})/(\lambda\sqrt{\pi})$.

Case 2

Moreover, if the energy eq. (1) is governed by a prescribed temperature boundary condition, namely:

$$T(0, t) = T_s, \quad t > 0, \quad T = T_\infty, \quad t > 0 \tag{4}$$

with exact solution [1]:

$$\frac{T(x,t) - T_\mathrm{s}}{T_s - T_\infty} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \quad q_\mathrm{s} = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \lambda\frac{T_s - T_\infty}{\sqrt{\pi}\sqrt{\alpha t}} \tag{5}$$

Then, the thermal impendence denoted as Z_2 is $Z_2 = \sqrt{\pi}(\sqrt{\alpha t}/\lambda)$. In both examples, due to the fixed position of the heat source, the transient resistance does not reach the static case [9, 15].

These simple cases only create the background of the problem developed in this work. The principle question in solving such problems is: why the entire domain solution $T(x, t)$ should to be developed and then setting $x = 0$, through $T(0, t)$ and $q(0, t)$, the thermal impedance has to be determined?

This work refers to determination of thermal impedances in 1-D semi-infinite problems due to constant and time-dependent heat sources acting on a half-plane area without development of entire domain solutions. The latter type of heat sources are results of runaway chemical reaction, uncontrolled nuclear processes and accidental flame effect on materials under fire. The main approach developed in this work is based on relationships between the local values of the heat gradient (flux) and the temperature based on fractional half time derivative and integrals [10, 13, 16, 17] without development the entire domain solutions. The fractional derivatives and integrals are not local [16] widely encountered in applications to transient rheology [18, 19], heat [10, 13], and mass transfer [20, 21], non-linear diffusion in porous and granular media [22], Stefan problem [23-25] manifest this technique as a power tool for efficient engineering solutions of complex problems.

Theoretical background: Fractional-time derivative and integrals

Following the results of Kulish and Lage [17] (see also ref. 13) the temperature and the flux governed by the model (1) are related by:

$$q(x,t) = \frac{\lambda}{\sqrt{\alpha}}\left[\frac{\partial^{1/2}T(x,t)}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}}\right] \tag{6a}$$

$$T(x,t) = \frac{\sqrt{\alpha}}{\lambda}\frac{\partial^{-1/2}[q(x,t)]}{\partial t^{-1/2}} + T_\infty \tag{6b}$$

The operators $\partial^{1/2}/t^{1/2}$ and $\partial^{-1/2}/t^{-1/2}$are half-time fractional derivative and integral in the Riemann-Liouville sense [16, 26], namely:

$$\mathrm{D}_t^{1/2} = \frac{\partial^{1/2} T(x,t)}{\partial_+ t^{1/2}} = \frac{1}{\Gamma(1/2)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \tag{7a}$$

$$\mathrm{D}_t^{-1/2} = \frac{\partial^{1/2} T(x,t)}{\partial_+ t^{1/2}} = \frac{1}{\Gamma(1/2)} \int_0^t \frac{T(x,u)}{\sqrt{t-u}} \mathrm{d}u \tag{7b}$$

were u is a dummy variable and $\Gamma(1/2) = \sqrt{\pi}$.

It is important to stress on the fact that the relationships (2.1a,b) come from the equation:

$$\frac{\partial T^{1/2}}{\partial t^{1/2}} = -\sqrt{\alpha} \frac{\partial T}{\partial x} \tag{8}$$

that can be obtained by representing eq. (1) as a product of two operators [16, 27] , namely:

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha} \frac{\partial}{\partial x} \right) \left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha} \frac{\partial}{\partial x} \right) T = 0 \tag{9}$$

Only the second term has physical meaning [16].The relationships (8) and (9) are non-local and valid over the entire semi-infinite domain.

Results: thermal impedance due stationary line sources

General relationships

From the general definitions of the thermal impedance we get for every x and t:

$$Z_1(x,t) = \frac{T(x,t) - T_\infty}{q(x,t)} = \frac{\dfrac{\sqrt{\alpha}}{\lambda} \dfrac{\partial^{-1/2}[q(x,t)]}{\partial t^{-1/2}}}{q(x,t)} \quad \text{with} \quad -\lambda \left. \frac{\partial T}{\partial z} \right|_{x=0} = q_s(t) \tag{10a}$$

$$Z_2(x,t) \equiv \frac{T(x,t) - T_\infty}{q(x,t)} \equiv \frac{T(x,t) - T_\infty}{\dfrac{\lambda}{\sqrt{\alpha}} \left[\dfrac{\partial^{1/2} T(x,t)}{\partial t^{1/2}} - \dfrac{T_\infty}{\sqrt{\pi t}} \right]} \quad \text{with} \quad T(0,t) = T_\mathrm{s}(t) \tag{10b}$$

Now, we will demonstrate how this approach works with some classical boundary conditions defining thermal loads at $x = 0$.

Time-independent boundary conditions at x = 0

Let us consider again the two simple cases that we commented at the beginning, where with $x = 0$ we get:

(1) with
$$-\lambda \left. \frac{\partial T}{\partial z} \right|_{x=0} = q_s = \text{const.} \tag{11a}$$

$$T(0,t) = \frac{\sqrt{\alpha}}{\lambda} \frac{\partial^{-1/2}[q(0,t)]}{\partial t^{-1/2}} + T_\infty = 2q_s \frac{\sqrt{\alpha t}}{\lambda \sqrt{\pi}} + T_\infty \tag{11b}$$

(2) with
$$T(0,t) = T_s = \text{const.} \tag{12a}$$

$$q_s = q(0,t) = \frac{\lambda}{\sqrt{\alpha}}\left[\frac{\partial^{1/2} T(0,t)}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}}\right] = \frac{\lambda}{\sqrt{\alpha t}}\frac{1}{\sqrt{\pi}}(T_s - T_\infty) \tag{12b}$$

Therefore,

$$Z_{1(\mathrm{frac})} = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\alpha t}}{\lambda}, \quad Z_{2(\mathrm{frac})} = \sqrt{\pi}\frac{\sqrt{\alpha t}}{\lambda} \tag{13a, b}$$

The results are the same as those obtained by the entire-domain solutions (3) and (5).

Time-depended line sources at x = 0

More interesting and practically cases emerge with time-dependent loads at $x = 0$. We consider some classic case widely used in transient heat transfer problems with many analytical and numerical solutions. This is the background of the results developed further in this work demonstrating the efficiency of the method.

Ramp (linear rise) thermal loads

Consider, two time-dependent analogues of the problems exemplified above.

Case 3

The thermal load at $x = 0$ is $T(0, t) = T_s = T_\infty + k_t t$.

Taking into account that $D_t^\mu(C) = [C/\Gamma(1-\mu)]t^{-\mu}$ and $D_t^\mu(Ct) = C^\mu[d^\mu(Ct)]/d(Ct)^\mu$ [16] for $\mu = 1/2$ we get:

$$q_{s(\mathrm{frac})}(0,t) = \frac{\lambda}{\sqrt{\alpha}}\left[\frac{\partial^{1/2}(T_\infty + k_t t)}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}}\right] = \frac{\lambda}{\sqrt{\alpha}}2\sqrt{k_t}\frac{\sqrt{t}}{\sqrt{\pi}} \tag{14}$$

$$Z_{2(T-\mathrm{ramp})} = \frac{k_t t}{2\dfrac{\lambda}{\sqrt{\alpha}}\sqrt{k_t}\dfrac{\sqrt{t}}{\sqrt{\pi}}} = \frac{\sqrt{\pi}\sqrt{k_t}}{2}\frac{\sqrt{\alpha t}}{\sqrt{\lambda}} \tag{15}$$

Case 4

Prescribed flux problem with a ramp rise $q(0, t) = q_s = k_q t$.

Further with $D_t^{-1/2}(C) = 2C\sqrt{t/\pi}$ and $D_t^{-1/2}(Ct) = (1/\sqrt{C})[d^{-1/2}(Ct)/d(Ct)^{-1/2}]$ [16] with $q(0, t) = q_s = k_q t$ the surface temperature:

$$T(0,t) = \frac{\sqrt{\alpha}}{\lambda}\frac{\partial^{-1/2}[q(0,t)]}{\partial t^{-1/2}} + T_\infty = \frac{4}{3}\frac{\sqrt{\alpha}}{\lambda}\frac{1}{\sqrt{\pi}}k_q\sqrt{t^3} + T_\infty \tag{16}$$

Then,

$$Z_{1(q-\mathrm{ramp})} = \frac{T(0,t) - T_\infty}{k_q T} = \frac{4}{3}\frac{1}{\sqrt{\pi}\sqrt{k_q}}\frac{\sqrt{\alpha t}}{\lambda} \tag{17}$$

Exponential temporal thermal sources at x = 0

Taking into account that [16]:

$$D_t^{1/2}\exp(k_e t) = \frac{1}{\sqrt{\pi t}} + \sqrt{k_e}\exp(k_e t)\,\mathrm{erf}\sqrt{k_e t} \tag{18a}$$

$$D_t^{-1/2}\exp(k_e t) = \frac{1}{\sqrt{k_e}}\exp(k_e t)\,\mathrm{erf}\sqrt{k_e t} \tag{18b}$$

$$D_t^{1/2}(Cf) = CD_t^{1/2}f, \quad D_t^{1/2}(Cf) = CD_t^{-1/2}f$$

Case 5

With $T(0, t) = T_\infty \exp(k_{et}t)$ we have:

$$q_{s(\text{frac})}(0,t) = \frac{\lambda}{\sqrt{\alpha}}\left[\frac{T_\infty}{\sqrt{\pi t}} + T_\infty\sqrt{k_{et}}\exp(k_{et}t)\operatorname{erf}\sqrt{k_{et}t}) - \frac{T_\infty}{\sqrt{\pi t}}\right] \tag{19a}$$

$$Z_{2(q-\exp)} = \frac{T_\infty \exp(k_{et}t)}{\dfrac{\lambda}{\sqrt{\alpha}}[T_\infty\sqrt{k_{et}}\exp(k_{et}t)\operatorname{erf}\sqrt{k_{et}t}]} = \frac{1}{\sqrt{k_{et}t}}\frac{1}{\operatorname{erf}\sqrt{k_{et}t}}\frac{\sqrt{\alpha t}}{\lambda} \tag{19b}$$

Case 6

Similarly with $q(0, t) = q_0\exp(k_{et}t)$, we have:

$$T(0,t) = \frac{\sqrt{\alpha}}{\lambda}\frac{\partial^{-1/2}[q(x,t)]}{\partial t^{-1/2}} + T_\infty = \frac{\sqrt{\alpha}}{\lambda}\left[q_0\frac{1}{\sqrt{k_{eq}}}\exp(k_{eq}t)\operatorname{erf}\sqrt{k_{eq}t}\right] + T_\infty \tag{20a}$$

$$Z_{2(T-\exp)} = \frac{q_0\dfrac{\sqrt{\alpha}}{\lambda}\left[\dfrac{1}{\sqrt{k_{eq}}}\exp(k_{eq}t)\operatorname{erf}\sqrt{k_{eq}t}\right]}{q_0\exp(k_{eq}t)} = \frac{\operatorname{erf}\sqrt{k_{eq}t}}{\sqrt{k_{eq}t}}\frac{\sqrt{\alpha t}}{\lambda} \tag{20b}$$

Periodic time-dependent boundary conditions x = 0

Consider now more complicated thermal loads [17] expressed by time-depended and varying functions, namely:

Case 7

With $T_s(t) = T_\infty + T_A\sin(\omega t)$ at $x = 0$ the boundary flux is:

$$q_s(t) = \frac{\lambda}{\sqrt{\alpha}}D_t^{1/2}[T_0 + T_A\sin(\omega t)] - \frac{\lambda}{\sqrt{\alpha}}\frac{T_\infty}{\sqrt{\pi t}} \tag{21a}$$

$$q_s(t) = \frac{\lambda}{\sqrt{\alpha}}T_A\sqrt{\omega}\left[\sin\left(\omega t + \frac{\pi}{4}\right) - \sqrt{2}\Lambda\sqrt{\frac{2\omega t}{\pi}}\right] \tag{21b}$$

and

$$Z_{2-T\sin(t)} = \frac{\sqrt{\alpha t}}{\lambda}\frac{\sin(\omega t)}{\sqrt{\omega t}\left[\sin\left(\omega t + \dfrac{\pi}{4}\right) - \sqrt{2}\Lambda\sqrt{\dfrac{2\omega t}{\pi}}\right]} \tag{22}$$

Case 8

With $q_s(t) = q_A\sin(\omega t)$ at $x = 0$ we have:

$$T_s(t) = \frac{\sqrt{\alpha}}{\lambda}D_t^{-1/2}[q_A\sin(\omega t)] = \frac{\sqrt{\alpha}}{\lambda}q_A\sqrt{\omega}\left[\sin\left(\omega t - \frac{\pi}{4}\right) + \sqrt{2}\Omega\sqrt{\frac{2\omega t}{\pi}}\right] \tag{23a}$$

$$Z_{1-q\sin(t)} = \frac{\sqrt{\alpha}}{\lambda}\frac{\sqrt{\omega}\left[\sin\left(\omega t - \dfrac{\pi}{4}\right) + \sqrt{2}\Omega\sqrt{\dfrac{2\omega t}{\pi}}\right]}{\sin(\omega t)} - \frac{T_\infty}{q_A\sin(\omega t)} \tag{23b}$$

Long-time behaviours

The auxiliary Fresnel function Λ[16, 28] in (21a) controls the initial transient regime. For long times enough to hold $\Lambda(t) \approx 0$ at, $t \sim 10\,\pi/\omega$ *i. e.* when max.$\left|\Lambda\sqrt{2\omega t/\pi}\right| < 10^{-3}$, we the medium attain the steady-periodic regime [17]. Hence (Case 7) the steady-periodic heat flux is $q_s(t) = (\lambda/\sqrt{\alpha})T_A\sqrt{\omega/2}$ [sin(ωt) + cos(ωt)]. Then, the steady-periodic thermal impedance $\bar{Z}_{2-T\sin(t)}$ is:

$$\bar{Z}_{2-T\sin(t)} = \frac{\sqrt{\alpha t}}{\lambda}\frac{\sin(\omega t)}{\sqrt{\frac{\omega t}{2}}[\sin(\omega t)+\cos(\omega t)]} \tag{24a}$$

Similarly, for the other auxiliary Fresnel function Ω [16, 28] in eq. (23a) becomes negligible at $t \sim 40\,\pi/\omega$, *i. e.* when max.$\left|\Omega\sqrt{2\omega/\pi}\right| < 4\cdot10^{-2}$ [17]. The corresponding steady-periodic thermal impedance is:

$$\bar{Z}_{1-q\sin(t)} = \frac{\sqrt{\alpha t}}{\lambda}\sqrt{\omega t}\left[\frac{\sin\left(\omega t - \frac{\pi}{4}\right)}{\sin(\omega t)}\right] - \frac{T_4}{q_A\sin(\omega t)} \tag{24b}$$

Other time-dependent thermal sources at x = 0

The results of two frequently encountered time-dependent boundary conditions are summarized in tab. 1 as good examples of the outcomes of the developed method.

Table 1. Thermal impedances and surface temperature/flux expressions for some time-dependent thermal loads

Function	Prescribed temperature	Flux	Prescribed flux	Temperature
$f = \sqrt{t}$	$T_s = T_\infty + b_t\sqrt{t}$	Eq. (6a) $q_s = \frac{\lambda}{\sqrt{\alpha}}\frac{b_t}{2}\sqrt{\pi}$	$q_s = b_q\sqrt{t}$	Eq. (6b) $T_s = \frac{\sqrt{\alpha}}{\lambda}\frac{b_q}{2}\sqrt{\pi t} + T_\infty$
$Z = \frac{\sqrt{\alpha t}}{\lambda}\frac{2}{\sqrt{\pi}}$			$Z = \frac{\sqrt{\alpha t}}{\lambda}\frac{\sqrt{\pi}}{2}$	
$f = t^2$	$T_s = T_\infty + c_t t^2$	$q_s = \frac{\lambda}{\sqrt{\alpha}}\left(c_t\frac{8\sqrt{t^3}}{3\sqrt{\pi}}\right)$	$q_s = c_q t^2$	$T_s = \frac{\sqrt{\alpha}}{\lambda}\frac{16}{15\sqrt{\pi}}c_q\sqrt{t^5} + T_\infty$
$Z = \frac{\sqrt{\alpha t}}{\lambda}\frac{3}{8}\sqrt{\pi}$			$Z = \frac{\sqrt{\alpha t}}{\lambda}\frac{16}{15\sqrt{\pi}}$	

Commentaries

The method developed here shows direct links between two interrelated issues required to calculate the transient thermal impedance of heated bodies. The common approach is to develop the entire domain solutions, then calculating the field gradient at $x = 0$ of the temperature at the same point is prescribed or *vice versa*. In most of cases, the entire domain solutions are cumbersome, needs time-wasting techniques, special function, *etc*. finally, and those solutions reduce to pre-factors depending on the imposed boundary conditions, when the space co-ordinate is set to zero. Hence, the fractional calculus approach avoids the development of entire-domain solutions because both the temperature and flux are related through fractional half time derivative and integral in the Riemann-Liouville sense.

The 1-D half-plane problems were especially chosen to demonstrate the approach because such problem are classic in the literature and in many cases either exact or approximate solutions exist, depending on the imposed boundary conditions, of course. Therefore, the results are directly comparable to those developed by the classical methods using entire-domain solutions. All results were especially present as products of $\sqrt{\alpha t}/\lambda$ which is a dimension of a "thermal impendence" [$KW^{-1}m^{-2}$] and terms depending on the type of the imposed boundary condition. This is a natural outcome of the basic Fourier law, because we have to mention that in semi-infinite problems with missing natural length scale, the role is played by $\sqrt{\alpha t}$ having a dimensions of length. Obviously, no limit can be attained since the area is infinite, too.

Cases of finite 1-D bodies can be also solved by fractional calculus methods but this needs different techniques and approaches beyond the scope of the present work.

Conclusions

The article presents a method for estimation and calculation of transient thermal impedances of 1-D semi-infinite areas (bodies) under various time-depending boundary conditions concerning both prescribed temperature and prescribed flux problem. The method is based on relationships based on Riemann-Liouville half-time fractional derivatives and integrals. The approach is simple and avoids development of entire-domain solutions.

Appendix

Fractional derivative-*definition and properties*

Fractional calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to 1/2 order. Despite "generalized" would be a better option, the name "fractional" is used for denoting this kind of derivative. The Riemann-Liouville derivative is the most used generalization of the derivative. It is based on Cauchy's formula for calculating iterated integrals (see refs.16 and 26 for more details):

$$\mathrm{D}^{\mu}\mathrm{f}(x) = \frac{1}{\Gamma(1-\mu)}\int_0^x \frac{\mathrm{f}(t)}{(x-t)^{\mu+1}}\mathrm{d}t \tag{A-1}$$

Fractional derivatives satisfy quite well all the properties that one could expect from them, despite some of them are only characteristic of integer order differentiation and some other have restrictions. Assuming further $\mu = 1/2$ for seek of clarity of the explanation in the main text we have:

– linearity
$$\mathrm{D}^{\mu}[a\mathrm{f}(x) + b\mathrm{g}(x)] = a\mathrm{D}\mu\mathrm{f}(x) + b\mathrm{D}^{\mu}\mathrm{g}(x) \tag{A-2}$$

– composition rule
$$\mathrm{D}^{\mu}\mathrm{D}^{\gamma}\mathrm{f}(x) = \frac{\mathrm{d}^{\mu+\gamma}}{\mathrm{d}x^{\mu+\gamma}}\mathrm{f}(x)$$

– with $\mu = 1/2$ (a semi-derivative) used in the work

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}C = \frac{Cx^{-1/2}}{\sqrt{\pi}} \tag{A-3}$$

$$\mathrm{D}^{1/2}\mathrm{f}(Cx) = C^{1/2}\frac{\mathrm{d}^{1/2}\mathrm{f}(Cx)}{\mathrm{d}(Cx)^{1/2}} \tag{A-4}$$

where C is any constant.

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Nomenclature

b_t, b_q – rate constant, [$Ks^{-1/2}$]
c_t – rate constant, [Ks^{-2}]
c_q – rate constant, [$Wm^{-2}s^{-2}$]
k_t, k_q, k_e, k_{et}, k_{eq} – rate constants, [s^{-1}]
q – heat flux density, [Wm^{-2}]
q_A – heat flux amplitude, [Wm^{-2}]
q_s – surface heat flux (x = 0), [Wm^{-2}]
T – temperature, [K]
T_A – temperature amplitude, [K]
T_∞ – temperature of the undisturbed medium, [K]
t – time, [s]
x – space co-ordinate, [m]
Z – thermal impedance, [Km^2W^{-1}]
Z_1 – thermal impedance of a prescribed flux problem, [Km^2W^{-1}]
Z_2 – thermal impedance of a prescribed temperature problem, [Km^2W^{-1}]
$Z_{1(\mathrm{frac})}$, $Z_{2(\mathrm{frac})}$ – thermal impedances developed by fractional derivative/ integral, [Km^2W^{-1}]

Greek symbols

α – thermal diffusivity, [m^2s^{-1}]
λ – thermal conductivity, [$Wm^{-1}K^{-1}$]
Γ – gamma function
Λ and Ω – auxiliary Fresnel integral (functions f and g in Abramowitz and Stegun [28]
ω – frequency, [s^{-1}]

Subscripts

s – surface (at x = 0)
T – prescribed temperature problem
q – prescribed flux problem

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