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LAPLACE TRANSFORM OVERCOMING PRINCIPLE DRAWBACKS IN APPLICATION OF THE VARIATIONAL ITERATION METHOD TO FRACTIONAL HEAT EQUATIONS

by

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This note presents a Laplace transform approach in the determination of the Lagrange multiplier when the variational iteration method is applied to time fractional heat diffusion equation. The presented approach is more straightforward and allows some simplification in application of the variational iteration method to fractional differential equations, thus improving the convergence of the successive iterations.

Key words: *variational iteration methods, Laplace transform, Lagrange multiplier, fractional heat diffusion equation*

Introduction

The application of the fractional calculus is a hot topic in heat transfer allowing solving many no-linear problems such as Stefan problem [1], the thermal sub-diffusion model [2], and the transition flows of complex fluids [3, 4]. Even though the fractional models are correctly describing non-linear real world phenomena, the solutions are quite complex and the real call among the scientists to find efficient analytical techniques for solutions of such problems in explicit forms.

The variational iteration method (VIM) [5, 6] is an analytical technique which has been widely used in the past ten year in non-linear problems. The key problem of the VIM is the correct determination of the Lagrange multiplier when the method is applied to fractional equations describing diffusion of heat or mass. This crucial point of the method is solved efficiently in the present work.

Problem formulation

The following integer-order parabolic equation describes transient heat conduction:

$$u_{\mathrm{t}} = cu_{\mathrm{xx}}, \quad u(0, x) = f(x) \tag{1}$$

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The solution in accordance to the VIM rules needs to construct the correction functional:

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)(u_{n,\tau} - cu_{n,xx})\mathrm{d}\tau, \quad u_0 = f(x) \tag{2}$$

The weighted function $\lambda(t, \tau)$ is called the Lagrange multiplier which can be determined by the variational theory looking for stationary conditions of the functional (2) [5, 6]. This procedure involves integration by parts of the integral in (2) that leads to a serious problem when the VIM is applied to differential equations of fractional order, namely:

$${}_0^C\mathrm{D}_t^\alpha u = cu_{xx}, \quad u(0,x) = f(x), \quad 0 < \alpha \leq 1 \tag{3}$$

Here ${}_0^C\mathrm{D}_t^\alpha$ is the Caputo derivative [7]. Equation (3) reduces to the classical one (1) for $\alpha = 1$.

Constructing the correction functional to eq. (3) we get:

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)({}_0^C\mathrm{D}_\tau^\alpha u_n - cu_{n,xx})\mathrm{d}\tau, \quad 0 < \alpha \tag{4}$$

The Lagrange multiplier in eq. (3) cannot be determined in a straightforward way like that in the integer-order model (1). The principle problem is the existence of the fractional derivative under the integral sign which does not allow the integration by parts to be performed. The section shows by applying the Laplace transform to eq. (3) the problem can be avoided.

Laplace transform approach in the Langrage multiplier's determination

Assuming for simplicity of the explanation $c = 1$ in eq.(3) and applying the Laplace transform L to both sides we get:

$$s^\alpha U(s) - u(0^+)s^{\alpha-1} = L[u_{xx}], \quad 0 < \alpha \leq 1 \tag{5}$$

where Laplace transform of the therm ${}_0^C\mathrm{D}_t^\alpha u$ holds [7]:

$$L[{}_0^C\mathrm{D}_t^\alpha u] = s^\alpha U(s) - \sum_{k=0}^{m-1} u^{(k)}(0^+)s^{\alpha-1-k}, \quad m = [\alpha]+1, \quad U(s) = L[u(t)] \tag{6}$$

Then, constructing the correction functional to (5) we have:

$$U_{n+1}(s) = U_n(s) + \lambda(s^\alpha U_n(s) - u(0,x)s^{\alpha-1} - L[u_{n,xx}]) \tag{7a}$$

The stationary condition of the functional (7a) require the following condition to be satisfied:

$$\frac{\delta U_{n+1}(s)}{\delta U_n(s)} = 0 \tag{7b}$$

This condition simply defines a Lagrange multiplier as:

$$\lambda = -\frac{1}{s^\alpha} \tag{7c}$$

As a result, the inverse Laplace transform of the variational iteration formula (7a) becomes:

$$\begin{cases} u_{n+1} = u_n + L^{-1}[\lambda(s^\alpha U_n(s) - u(0,x)s^{\alpha-1} - L[u_{n,xx}])] = u_0 + L^{-1}\left[\dfrac{1}{s^\alpha}(L[u_{n,xx}])\right] \\ u_0 = u(0,x) = f(x) \end{cases} \tag{8}$$

On the other hand, we recently give another way to identify the Lagrange multiplier in [8-10]:

$$\begin{cases} u_{n+1} = u_n + \int\limits_0^t \lambda(t,\tau)({}_0^C\mathrm{D}_\tau^\alpha u_n - u_{n,xx})\mathrm{d}\tau, \quad 0<\alpha \\ \lambda(t,\tau) = \dfrac{(-1)^\alpha(\tau-t)^{\alpha-1}}{\Gamma(\alpha)} \end{cases} \tag{9a, b}$$

Both (9a, 9b) and (8) can lead to the same result.

The Lagrange multiplier (9b) transforms the Riemann integral (9a) of the iteration functional into the Riemann-Liouville (R-L) integral. This is a correct approach because if we apply to the R-L integral ${}_0I_t^\alpha$ both sides of eq. (3), than the correct iteration formula should be:

$$u_{n+1} = u_n + {}_0I_t^\alpha[\lambda(t,\tau)({}_0^C\mathrm{D}_\tau^\alpha u_n - u_{n,xx})], \quad 0<\alpha \tag{10}$$

However, albeit the correctness of (10) the impossibility to apply the integration by parts in the fractional integral led to a simplification by replacing it by the Riemann integral as it defined by (9a). The simplification ever continued with the Lagrange multiplier as $\lambda = -1$ [11-14]. This chain of simplifications leads to a poor convergence of the iteration formula. The Laplace transform approach in the determination of $\lambda(t,\tau)$ corrects the second step of the simplification chain and results in what the iteration formula should be. The final result is, in fact, the Lagrange multiplier defined by (9b) is the kernel of the R-L integral. This point was analyzed recently by Hristov [15] with two options in the integration: (1) iteration formula as it is defined by (9a, b) and (2) iteration formula with $\lambda = -1$ and the R-L integral defined by (10).

Example: Fractional heat diffusion equation of the R-L type

In this section, we apply our method to the fractional heat-diffusion equation of the R-L type, namely:

$${}_0^{RL}\mathrm{D}_t^\alpha u = u_{xx}, \quad {}_0I_t^{1-\alpha}u(0^+) = \sin(x), \quad 0<\alpha\le 1 \tag{11}$$

which can also describes a transient flow in a porous medium.

Our approach leads to the following iteration formula:

$$\begin{cases} u_{n+1} = u_0 + L^{-1}\left[\dfrac{1}{s^\alpha}(L[u_{n,xx}])\right] \\ u_0 = \dfrac{\sin(x)t^{\alpha-1}}{\Gamma(\alpha)} \end{cases}$$

The successive iterations are obtained as:

$$\begin{aligned} u_1 &= \frac{\sin(x)t^{\alpha-1}}{\Gamma(\alpha)} - \frac{\sin(x)t^{2\alpha-1}}{\Gamma(\alpha+\alpha)} \\ &\vdots \\ u_n &= \sin(x)t^{\alpha-1}\sum_{k=0}^{n}\frac{(-t)^{k\alpha}}{\Gamma(\alpha+k\alpha)} \end{aligned} \tag{12}$$

For $n\to\infty$, u_n rapidly tends to $\sin(x)t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)$ which is an exact solution of (11). The diffusion behaviors are shown in fig. 1 at different fractional orders.

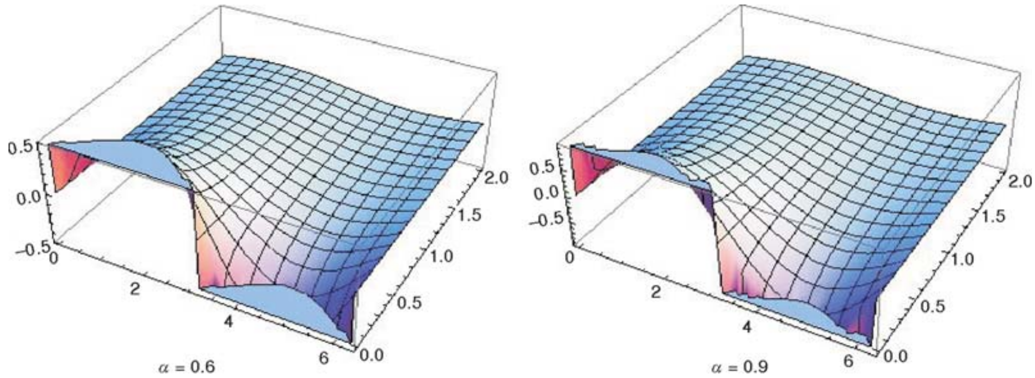

Figure 1. VIM solutions to the sub-diffusion equation at various α(0 < $\alpha \leq$ 1)

Conclusions

This scientific note presents the application of Laplace transform in correct determination of Lagrange multiplier when the VIM is applied to fractional heat-diffusion equations. The approach is exemplified by solutions of fractional heat diffusion equations with the Caputo derivative and the R-L derivative, respectively. The results show that the new approach is more efficient and straightforward to identify the Lagrange multiplier here and yet give approximate solutions of high accuracies. The VIM now can be a reliable tool to analytically investigate heat models with fractional derivatives.

Nomenclature

c	– specific heat capacity, [Jkg^{-1}]
${}_0^C D_t^\alpha u$	– time-fractional Caputo derivative
${}_0^{RL} D_t^\alpha u$	– time-fractional Riemann-Liouville derivative
$E_{\alpha,\beta}$	– Mittag-Leffler function with parameters α and β
${}_0I_t^\alpha$	– Riemann-Liouville integral of α order
L	– Laplace transform
m	– integer between α and α + 1
n	– order of the approximate solutions
s	– complex argument of Laplace transform
t	– time, [s]
U_s	– Laplace transform of $u(t)$
u	– temperature, [K]
u_n	– n-th order approximate solution
x	– space co-ordinate, [m]

Greek symbols

α	– fractional order [–]
δ	– variation operator
Γ	– gamma function
$\lambda(t, \tau)$	– Lagrange multiplier
τ	– time, [s]

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