HEAT TRANSFER FOR TWO TYPES OF VISCOELASTIC FLUID OVER AN EXPONENTIALLY STRETCHING SHEET WITH VARIABLE THERMAL CONDUCTIVITY AND RADIATION IN POROUS MEDIUM

by

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Original scientific paper
DOI: 10.2298/TSCI111102144S

An analysis has been carried out to study the boundary layer flow and heat transfer characteristics of second order fluid and second grade fluid with variable thermal conductivity and radiation over an exponentially stretching sheet in porous medium. The basic boundary layer equations governing the flow and heat transfer in prescribed surface temperature and prescribed heat flux cases are in the form of partial differential equations. These equations are converted to non-linear ordinary differential equations using similarity transformations. Numerical solutions of the resulting boundary value problem are solved by using the fourth order Runge-Kutta method with shooting technique for various values of the physical parameters. The effect of variable thermal conductivity, porosity, Prandtl number, radiation parameter, and viscoelastic parameters on velocity and temperature profiles (in prescribed surface temperature and prescribed heat flux cases) are analyzed and discussed through graphs. Numerical values of wall temperature gradient in prescribed surface temperature case and wall temperature in prescribed heat flux case are obtained and tabulated for various values of the governing parameters. In this study Prandtl number also treated as variable inside the boundary layer because it depends on thermal conductivity. The results are also verified by using finite difference method.

Key words: *second grade fluid, second order fluid, porous medium, exponentially stretching sheet, radiation*

Introduction

Aerodynamic extrusion of plastic sheets, glass fiber production, paper production, heat treated materials traveling between a feed roll and a wind-up roll, cooling of an infinite metallic plate in a cooling bath, manufacturing of polymeric sheets are some examples for practical applications of non-Newtonian fluid flow over a stretching surface. For the production of fiber sheet/plastic sheet, extrusion of molten polymers through a slit die is an important process in polymer industry. This thermo-fluid problem involves significant heat transfer between the sheet and the surrounding fluid. In this process the extrudate starts to solidify as soon as it exits

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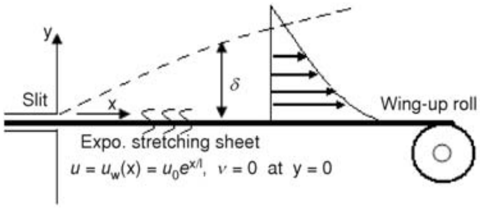

Figure 1. Physical model of the problem

from the die and than sheet is collected by a wind-up roll upon solidification (fig. 1). The quality of the final product depends on the rate of heat transfer at the stretching surface. This stretching may not necessarily linear. It may be quadratic, power-law, exponential and so on. After the pioneering work of Sakiadis [1, 2] many researcher gave attention to study heat transfer of Newtonian and non-Newtonian fluids over a linear stretching sheet. By considering quadratic stretching sheet, Kumaran and Ramanaiah [3] analyzed the problem of heat transfer. Ali [4] investigated the thermal boundary layer flow on a power law stretching surface with suction or injection. Elbashbeshy [5] analyzed the problem of heat transfer over an exponentially stretching sheet with suction. Magyari and Keller [6] discussed the heat and mass transfer in boundary layers on an exponentially stretching continuous surface. Sanjayanand and Khan [7, 8] extended the work of Elbashbeshy [5] to viscoelastic fluid flow, heat and mass transfer over an exponentially stretching sheet.

To further improve the mechanical properties of the fiber sheet/plastic sheet, it is important to control its rate of cooling. Mainly the rate of cooling depends on physical properties of cooling medium *e. g.* its thermal conductivity, radiative heat transfer property of cooling medium and porous medium. Water is the most widely used fluid to be used as the cooling medium. To have a better control on the rate of cooling we have to control its viscoelasticity by using polymeric additives [9]. By using such additives the viscosity of the fluid is increased and it slows down the rate of solidification.

The transport of heat in a porous medium has considerable practical applications in geothermal systems, crude oil extraction, and ground water pollution and also in a wide range of bio mechanical problems. The flow of a steady viscous fluid and heat transfer characteristics in a porous medium by considering different heating processes is studied by Vajravelu [10]. The problem for viscoelastic fluid flow and heat transfer in a porous medium over a stretching sheet studied by Subhas and Veena [11]. The solution for both heat and mass transfer in hydromagnetic flow of a non-Newtonian fluid with heat source over an accelerated surface through porous medium has found by Eldabe and Mohamed [12].

Radiative heat transfer flow is very important in manufacturing industries for the design of reliable equipments, nuclear plants, gas turbines and various propulsion devices for missiles, aircraft, satellites and space vehicles. Also, the effects of thermal radiation on the forced and free convection flows are important in the context of space technology and processes involving high temperature. Cogley *et al.* [13] showed that with in the optically thin limit, the fluid does not absorb its own emitted radiation but the radiation emitted by the boundary is observed by the fluid. Many of the researchers have considered the effect of radiation on flows involving a viscoelastic fluid. Raptis [14, 15], Raptis and Perdikis [16], Siddheshwar *et al.* [17], Khan [18], found the effect of radiation on heat transfer by considering different viscoelastic fluids, heat source/sink, suction/blowing over a stretching sheet. Firstly, Sajid and Hayet [19] discussed the influence of thermal radiation on the boundary layer flow due to an exponentially stretching sheet. They used homotopy analysis method (HAM) to solve the problem analytically.

All these investigations are carried out taking into account of constant physical properties of the ambient fluid but practical situations demand for physical properties with variable characteristics. Thermal conductivity is one of such properties. In general, the thermal conductivity is strongly temperature dependent or thermal conductivity is assumed to vary linearly with temperature. Abel *et al.* [20] have considered the effect of variable thermal conductivity with temperature dependent heat source/sink, in presence of thermal radiation. Chiam [21, 22] studied the effect of variable thermal conductivity on heat transfer. Abel and Mahesha [23] have investigated the heat transfer in MHD viscoelastic fluid flow over a stretching sheet with variable thermal conductivity, non-uniform heat source and radiation. Rahman and Salahuddin [24] have investigated the MHD heat and mass transfer flow over a radiative isothermal inclined heated surface with variable viscosity and electric conductivity.

Chen [25] has obtained the analytical solution of MHD flow and heat transfer of an electrically conducting two types of viscoelastic fluid past a stretching surface with internal heat generation/absorption and thermal radiation. In his study he also considered work done due to deformation, joule and viscous dissipation.

To the best of the author's knowledge, not much work has been done on the consideration of the effect of variable thermal conductivity on the two types of flows (second order fluid and second grade fluid) over an exponentially stretching sheet through porous medium. The main aim of this paper is to study the effect of viscoelastic parameter, radiation parameter, porosity parameter, variable thermal conductivity parameter and variable Prandtl number on the flow and heat transport in PST and PHF cases graphically.

Basic equation

The constitutive equation for an incompressible homogeneous non-Newtonian fluid is:

$$\mathbf{T} = -P_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{1}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, P_1 is the pressure, $\mathbf{I}$ – the identity tensor, μ – the dynamic viscosity and α_1, and α_2 are the normal stress moduli. $\mathbf{A}_1$ and $\mathbf{A}_2$ are the Rivlin-Ericksen [26] tensors given by:

$$\mathbf{A}_1 = \text{grad}\vec{\text{v}} + (\text{grad}\vec{\text{v}})^T \tag{2}$$

$$\mathbf{A}_2 = \frac{\text{d}\mathbf{A}_1}{\text{d}t} + (\text{grad}\vec{\text{v}})^T\mathbf{A}_1 + \mathbf{A}_1(\text{grad}\vec{\text{v}}) \tag{3}$$

In the equations $\vec{\text{v}}$ is the velocity, grad denotes the gradient operator and d/dt denote the material time derivative. Equation (1) was derived by Coleman and Noll [27] using the postulates of gradually fading memory. This equation has invariant property so it has been considered as an exact model for some fluids. According to Dunn and Fosdick [28], the second-order fluid model is compatible with thermodynamics when the Helmholtz free energy of the fluid is a minimum for the fluid in equilibrium. They found that the material moduli must satisfy:

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + a_2 = 0 \tag{4}$$

Fosdick and Rajagopal [29] have shown that the material moduli μ, α_1, and α_2 should satisfy the following relations in case of second order fluid:

$$\mu \geq 0, \quad \alpha_1 \leq 0, \quad \alpha_1 + \alpha_2 \neq 0 \tag{5}$$

Generally, in the literature the fluid satisfied the model (1) with $\alpha_1 < 0$ is termed as second order fluid and with $\alpha_1 > 0$ is termed as second grade fluid. Equation (1) reduces the constitutive relation of an incompressible Newtonian fluid when we take $\alpha_1 = 0, \alpha_2 = 0$ and $\mu > 0$. Another class of models is the rate-type fluid models, such as Walters' liquid B model, which represents an approximation to the first order in elasticity *i. e.* for short or rapidly fading memory fluids. Beard and Walters [30] derived the equations for Walters' liquid B.

Consider the steady 2-D boundary layer flow of an incompressible, viscoelastic fluid past a stretching sheet coinciding with the plane $y = 0$ (see fig. 1). In formulating the problem we consider the assumptions [7]:

– the boundary sheet is assumed to be moving axially with a velocity of exponential order in the axial direction and generating the boundary layer type of flow, and
– the normal stress is of the same order of magnitude as that of the shear stress, in addition to the usual boundary layer approximations.

Under the above considerations of the problem, the conservation equations of mass and momentum for the flow of viscoelastic fluid can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\kappa_0\left(u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)-\frac{\nu u}{k'} \tag{7}$$

where u and v are the velocity components in the x- and y-directions, respectively, ν – the coefficient of kinematic viscosity, and $\kappa_0 = -\alpha/\rho$ – the elastic parameter. It is noted as $\kappa_0 > 0$ is for second order fluid and $\kappa_0 < 0$ indicates Walters' liquid B also termed as second grade fluid and $\kappa_0 = 0$ denote the incompressible Newtonian fluid. k' is the permeability of the porous media.

Consider the initial and boundary conditions on velocity:

$$u = u_w(x) = u_0 \mathrm{e}^{x/l} \quad \text{at } y=0, \quad v=0 \quad \text{at } y=0, \quad u=0 \quad \text{at } y\to\infty \tag{8}$$

where u_0 is a constant and l is the reference length.

Solution of momentum boundary layer equation

The velocity component u and v in terms of stream function $\psi(x, y)$ can be written as:

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x} \tag{9}$$

For solving momentum equation, introduce a similarity variable η, such that:

$$\eta = y\sqrt{\frac{u_0}{2\nu l}}\mathrm{e}^{x/2l} \tag{10}$$

$$\psi(x, y) = \sqrt{2\nu l u_0}\, f(x,\eta)\mathrm{e}^{x/2l} \tag{11}$$

Here f is the dimensionless stream function and considering $f(x, \eta) = f(\eta)$ [8]. Making use of (9)-(11) in eq. (7) we obtain a fourth order non-linear ordinary differential equation of the form:

$$2f_\eta^2 - ff_{\eta\eta} = f_{\eta\eta\eta} - k_1^*\left(3f_{\eta\eta} - \frac{1}{2}ff_{\eta\eta\eta\eta} - \frac{3}{2}f_\eta^2\right) - 2Rf_\eta \tag{12}$$

where $k_1^* = k_0 u_w/\nu l$ is the local viscoelastic parameter and $R = \nu l/u_w k'$ – the porosity parameter. It is noted that $k_1^* > 0$ for second order fluid and $k_1^* < 0$ for second grade fluid. The boundary conditions on f are:

$$f=0, \quad f_\eta=1 \quad \text{at} \quad \eta=0, \quad f_\eta=0 \quad \text{as} \quad \eta\to\infty \tag{13}$$

Integrating eq. (12)

$$f_{\eta\eta}+ff_\eta=\alpha+2Rf+\int_0^\eta\left[3f_{\eta_1}^2+k_1^*\left(3f_{\eta_1}f_{\eta_1\eta_1\eta_1}-\frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1}-\frac{3}{2}f_{\eta_1\eta_1}^2\right)\right]\mathrm{d}\eta_1 \tag{14}$$

for $\eta\to\infty$, we obtain:

$$\alpha=-\int_0^\infty\left[3f_{\eta_1}^2+k_1^*\left(3f_{\eta_1}f_{\eta_1\eta_1\eta_1}-\frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1}-\frac{3}{2}f_{\eta_1\eta_1}^2\right)\right]\mathrm{d}\eta_1 \tag{15}$$

Again integrating eq. (14) and applying boundary conditions, we obtain:

$$f_\eta+\frac{f^2}{2}=\alpha\eta+\int_0^\eta\left\{\int_0^{\eta_2}\left[3f_{\eta_1}^2+k_1^*\left(3f_{\eta_1}f_{\eta_1\eta_1\eta_1}-\frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1}-\frac{3}{2}f_{\eta_1\eta_1}^2\right)\right]\mathrm{d}\eta_1\right\}\mathrm{d}\eta_2+2R\int_0^\eta\mathrm{d}\eta+1 \tag{16}$$

Assume zeroth-order approximation of $f_\eta^0(\eta)$ as:

$$f_\eta^{(0)}(\eta)=\exp(-\alpha_0\eta), \quad \alpha_0>0 \tag{17}$$

which satisfying the boundary conditions (13). Integrating eq. (17) and making use of boundary conditions at $\eta=0$ of the eq. (13) we get:

$$f^{(0)}(\eta)=\frac{1-\exp(-\alpha_0\eta)}{\alpha_0} \tag{18}$$

where

$$\alpha_0=\sqrt{\frac{3+4R}{2(1-k_1^*)}} \tag{19}$$

The solution procedure of the eq. (16) may be reduced to the sequential solution of the Riccati-type equations:

$$f_\eta^{(n)}+\frac{1}{2}f^{(n)2}=RHS\left(f_\eta^{(n-1)},f_{\eta\eta}^{(n-1)},f_{\eta\eta\eta}^{(n-1)},f_{\eta\eta\eta\eta}^{(n-1)}\right) \tag{20}$$

The equation for first order iteration $f_\eta^{(1)}(\eta)$ takes the form:

$$f_\eta^{(1)}+\frac{1}{2}f^{(1)2}=\left(\frac{3+k_1^*\alpha_0^2}{4\alpha_0^2}\right)(\mathrm{e}^{-2\alpha_0\eta}-1)+\left[\frac{k_1^*}{2}+\frac{2R}{\alpha_0^2}\right](\mathrm{e}^{-\alpha_0\eta}-1)+\frac{2R\eta}{\alpha_0}+1 \tag{21}$$

Equation (21) is a non-linear Riccati-type equation which for $f^{(1)}(\eta)$ can be solved analytically. However we use zeroth-order approximation $f^{(0)}(\eta)$ for solving the energy equation.

The dimensionless skin friction coefficient c_f is expressed as:

$$c_f=-\frac{\left[\nu\dfrac{\partial u}{\partial y}-\kappa_0\left(u\dfrac{\partial^2 u}{\partial x\partial y}-2\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}\right)\right]}{u_0^2\exp\left(\dfrac{2x}{l}\right)}=-\frac{\alpha_0}{\sqrt{2\,\mathrm{Re}}}\left(1-\frac{7}{2}k_1^*\right) \tag{22}$$

Here, $\mathrm{Re}=u_\mathrm{w}l/\nu$ is the Reynolds number.

Heat transfer analysis

The governing boundary layer heat transport equation with variable thermal conductivity and radiation (see fig. 1) is given by:

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{\partial q_r}{\partial y} \tag{23}$$

where T is the temperature of the fluid, ρ – the density of the fluid, and c_p – the specific heat at constant pressure.

The thermal conductivity k is assumed to vary linearly with temperature [23] and it is of the form:

$$k=\begin{Bmatrix} k_\infty[1+\varepsilon\theta(\eta)] \text{ in PST case} \\ k_\infty[1+\varepsilon\theta(\eta)] \text{ in PHF case}\end{Bmatrix} \tag{24}$$

where ε is the small parameter which is negative for most solids and liquids and positive for gases [31], $\theta(\eta)$ – a dimensionless scaled temperature in PST case, and $\phi(\eta)$ – the non-dimensional scaled temperature in PHF case.

The radiative heat flux q_r is modeled as:

$$q_r=-\frac{4\sigma_1}{3m}\frac{\partial(T^4)}{\partial y} \tag{25}$$

where σ_1 is the Stefan-Boltzmann constant and m – the mean absorption coefficient. Assuming that the difference in temperature within the flow is such that T^4 can be expressed as a linear combination of temperature, we expand T^4 in a Taylor series about T_∞ as:

$$T^4=T_\infty^4+4T_\infty^3(T-T_\infty)+6T_\infty^2(T-T_\infty)^2+\cdots \tag{26}$$

and neglecting higher order terms beyond the first degree in $(T-T_\infty)$, we get:

$$T^4\cong -3T_\infty^4+4T_\infty^3T \tag{27}$$

Substituting eqs. (27) in (26) we obtain:

$$\frac{\partial q_r}{\partial y}=-\frac{16T_\infty^3\sigma_1}{3m}\frac{\partial^2 T}{\partial y^2} \tag{28}$$

Using eqs. (28) in (23) we obtain:

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left[\left(k+\frac{16T_\infty^3\sigma_1}{3m}\right)\frac{\partial T}{\partial y}\right] \tag{29}$$

The thermal boundary conditions for solving eq. (29) depend on the type of heating process to be considered. We employ the following two types of heating processes:

– prescribed surface temperature (PST), and
– prescribed heat flux (PHF).

PST case: In this case the boundary conditions are of the form:

$$T=T_\mathrm{w}=T_\infty+A\mathrm{e}^{ax/sl} \quad \text{at} \quad y=0, \quad T\to T\infty \quad \text{as} \quad y\to\infty \tag{30}$$

where A and a are the parameters of temperature distribution depending on the properties of the liquid. Define the non-dimensional temperature parameter $\theta(\eta)$ in this case as:

$$\theta(\eta)=\frac{T-T_\infty}{T_\mathrm{w}-T_\infty} \tag{31}$$

where $T-T_\infty=A\mathrm{e}^{ax/2l}, \theta(\eta)$ and $T_\mathrm{w}-T_\infty=A\mathrm{e}^{ax/2l}$.

Using eq. (31) in eq. (29), we obtain a non-linear ordinary differential equation for $\theta(\eta)$ in the form:

$$(1+Tr+\varepsilon\theta)\theta'' + \Pr_\infty f\theta' - a\Pr_\infty f\theta + \varepsilon\theta'^2 = 0 \tag{32}$$

where $\Pr_\infty = \dfrac{\mu c_p}{k_\infty}$ is the ambient Prandtl number at $\eta \to \infty$ (33)

and $Tr = \dfrac{16\sigma_1 T_\infty^3}{3k_\infty k_1^*}$ is the thermal radiation parameter. (34)

As a particular case we take $a = 2$ and obtain the equation:

$$(1+Tr+\varepsilon\theta)\theta'' + \Pr_\infty f\theta' - 2\Pr_\infty f'\theta + \varepsilon\theta'^2 = 0 \tag{35}$$

Corresponding thermal boundary condition become:

$$\theta(0) = 1, \quad \theta(\infty) \to 0 \tag{36}$$

PHF case: The boundary conditions in case of exponential order heat flux are of the form:

$$-k_\infty\left(\frac{\partial T}{\partial y}\right)_\mathrm{w} = B\mathrm{e}^{\left(\frac{b+1}{2l}\right)x} \quad \text{at} \quad y=0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty \tag{37}$$

where B and b are the parameters of temperature distribution depending on the properties of the fluid and $k = k_\infty[1+\varepsilon\phi(\eta)]$. Now we define the non-dimensional temperature parameter $\phi(\eta)$ as:

$$\phi(\eta) = \frac{T - T_\infty}{\dfrac{B}{k_\infty}\sqrt{\dfrac{2vl}{u_0}}\mathrm{e}^{bx/2l}} \tag{38}$$

Using eq. (38) in eq. (29) we obtain the non-linear ordinary differential equation for $\phi(\eta)$ in the form:

$$(1+Tr+\varepsilon\phi)\phi'' + \Pr_\infty f\phi' - b\Pr_\infty f'\phi + \varepsilon\phi'^2 = 0 \tag{39}$$

Corresponding boundary condition for $\phi(\eta)$ are given by:

$$\phi'(\eta) = -\frac{1}{1+\varepsilon} \quad \text{at} \quad \eta = 0, \quad \phi(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{40}$$

As a particular case we assign $b = 2$ and obtain the following equation:

$$(1+Tr+\varepsilon\phi)\phi'' + \Pr_\infty f\phi' - 2\Pr_\infty f'\phi + \varepsilon\phi'^2 = 0 \tag{41}$$

Investigation of flow behavior and heat transfer would be carried out by analyzing the skin friction coefficient and Nusselt number at the wall which are proportional to the numerical values of $f''(0)$, $-\theta'(0)$ in PST case and $1/\phi(0)$ in PHF case, respectively.

Variable Prandtl number

The Prandtl number is a function of thermal conductivity and viscosity. Since the thermal conductivity is assumed to vary linearly with temperature across the boundary layer, the Prandtl number varies too. The assumption of constant Prandtl number inside the boundary layer produces unrealistic results [32]. Therefore, Prandtl number related to the variable thermal conductivity is defined by:

$$\mathrm{Pr} = \frac{\mu c_p}{k} \tag{42}$$

in PST case:

$$\mathrm{Pr} = \frac{\mu c_p}{k_\infty [1 + \varepsilon\theta(\eta)]}$$

$$\mathrm{Pr}_\infty = \mathrm{Pr}[1 + \varepsilon\theta(\eta)] \tag{43}$$

similarly in PHF case:

$$\mathrm{Pr}_\infty = \mathrm{Pr}[1 + \varepsilon\phi(\eta)] \tag{44}$$

using eqs. (43) in (35) and (44) in (41) the energy equation in PST and PHF case can be written as:

$$(1 + Tr + \varepsilon\theta)\theta'' + \mathrm{Pr}(1 + \varepsilon\theta)f\theta' - 2\mathrm{Pr}(1 + \varepsilon\theta)f'\theta + \varepsilon\theta'^2 = 0 \tag{45}$$

$$(1 + Tr + \varepsilon\phi)\phi'' + \mathrm{Pr}(1 + \varepsilon\phi)f\phi' - 2\mathrm{Pr}(1 + \varepsilon\phi)f'\phi + \varepsilon\phi'^2 = 0 \tag{46}$$

These equations are the corrected non-dimensional form of the energy equation in PST and PHF form in which Prandtl number is treated as variable. It can be seen that $\mathrm{Pr} \to \mathrm{Pr}_\infty$ as $\eta \to \infty$. In that case eqs. (45) and (46) reduces to eqs. (35) and (41), respectively.

Numerical procedure

Runge-Kutta method

Equations (12) and (45) constitute a highly non-linear coupled boundary value problem of fourth order in f and second order in θ, respectively. These equations are solved by numerically using shooting technique with fourth order Runge-Kutta integration algorithm. The coupled boundary value problem (12) and (45) has been reduced to a system of six simultaneous ordinary differential equations of first-order for six unknowns following the method of superposition by assuming $f = f_1$, $f' = f_2$, $f'' = f_3$, $f''' = f_4$, $\theta = \theta_1$, $\theta' = \theta_2$. To solve this system of equations we require six initial conditions whilst we have only two initial conditions $f(0)$ and $f'(0)$ on f and one initial condition $\theta(0)$ on θ. The third initial condition on $f''(0)$ on f has been deduced by applying initial conditions given by eqs. (13) in (12). Still there are two initial conditions $f''(0)$ and $\theta'(0)$ which are not prescribed, however, the values of $f'(\eta)$ and $\theta(\eta)$ are known at $\eta \to \infty$. For employing shooting technique, to select η_∞ we begin with the initial approximation as $f_3(0) = \alpha_0$ and $\theta_2(0) = \beta_0$. Let α and β be correct values of $f_3(0)$ and $\theta_2(0)$, respectively. After solving the system of six differential equations using fourth order Runge-Kutta method and finding the values of $f_3(0) = f_3(\alpha_0, \beta_0, \eta_\infty)$ and $\theta_2(0) = \theta_2(\alpha_0, \beta_0, \eta_\infty)$ at $\eta = \eta_\infty$. The solution process repeated with another larger value of η_∞ until two successive values of $f_3(0)$ and $\theta_2(0)$ differs only after desired digit signifying the limit of boundary along η. The last value of η_∞ is chosen as appropriate value for that particular set of parameters. Finally, the problem has been solved numerically using fourth order Runge-Kutta integration scheme. In all the computations the step size $\eta = 0.001$ was selected that satisfied a convergence criterion of 10^{-5} in almost all of different phase mentioned above. The maximum value of $\eta_\infty = 25$ is taken in this problem. Similarly we solve eqs. (12) and (46) by using same technique described.

Finite difference method

The momentum eq. (12) and energy eqs. (45) in PST case and (46) in PHF case are also solved by using finite difference method. Firstly for solving momentum equation, which is fourth order non-linear non-homogeneous differential equation, linearization technique is applied to convert the non-linear terms to a linear stage. Then, the implicit finite difference method is used to replace the different terms by their second-order central difference approximation:

$$2\left(\frac{\bar{f}_{i+1}-\bar{f}_{i-1}}{2\eta}\right)\left(\frac{f_{i+1}-f_{i-1}}{2\eta}\right)-\bar{f}_i\left(\frac{f_{i+1}-2f_i+f_{i-1}}{\eta^2}\right)=\left(\frac{f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2}}{2\eta^3}\right)-$$
$$-3k_1^*\left(\frac{\bar{f}_{i+1}-\bar{f}_{i-1}}{2\eta}\right)\left(\frac{f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2}}{2\eta^3}\right)+$$
$$+k_1^*\frac{\bar{f}_i}{2}\left(\frac{f_{i+2}-4f_{i+1}+6f_i-4f_{i-1}+f_{i-2}}{\eta^4}\right)+$$
$$+\frac{3}{2}k_1^*\left(\frac{\bar{f}_{i+1}-2\bar{f}_i+\bar{f}_{i-1}}{\eta^2}\right)\left(\frac{\bar{f}_{i+1}-2\bar{f}_i+\bar{f}_{i-1}}{\eta^2}\right)-2R\left(\frac{f_{i+1}-f_{i-1}}{2\eta}\right) \tag{47}$$

with the boundary conditions

$$f_0=0,\quad \frac{f_1-f_{-1}}{2\eta}=1,\quad \frac{f_{n-1}-f_{n+1}}{2\eta}=0 \tag{48}$$

After using boundary conditions in eq. (47) we get tridiagonal system of equations which are solved by using Thomas algorithm to obtain $f(\eta)$. The same technique as described can also be adopted to solve energy eqs. (45) in PST case and (46) in PHF case. The finite difference approximations for energy equations in PST case:

$$(1+Tr+\varepsilon\bar{\theta}_i)\left(\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{\eta^2}\right)+\Pr_\infty(1+\varepsilon\bar{\theta}_i)\left[\bar{f}_i\left(\frac{\theta_{i+1}-\theta_{i-1}}{2\eta}\right)-2\bar{\theta}_i\left(\frac{f_{i+1}-f_{i-1}}{2\eta}\right)\right]+$$
$$+\varepsilon\left(\frac{\bar{\theta}_{i+1}-\bar{\theta}_{i-1}}{2\eta}\right)\left(\frac{\theta_{i+1}-\theta_{i-1}}{2\eta}\right)=0 \tag{49}$$

with thermal boundary conditions:

$$\theta_0=1,\quad \theta_n=0 \tag{50}$$

and in PHF case:

$$(1+Tr+\varepsilon\bar{\phi}_i)\left(\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2}\right)+\Pr_\infty(1+\varepsilon\bar{\phi}_i)\left[\bar{f}_i\left(\frac{\phi_{i+1}-\phi_{i-1}}{2h}\right)-2\bar{\phi}_i\left(\frac{f_{i+1}-f_{i-1}}{2h}\right)\right]+$$
$$+\varepsilon\left(\frac{\bar{\phi}_{i+1}-\bar{\phi}_{i-1}}{2h}\right)\left(\frac{\bar{\phi}_{i+1}-\bar{\phi}_{i-1}}{2h}\right)=0 \tag{51}$$

with thermal boundary conditions:

$$\frac{\phi_1-\phi_{-1}}{2h}=-\frac{1}{1+\varepsilon},\quad \phi_n=0 \tag{52}$$

The variables with bars are given initial guesses from the previous steps. The resulting system of equations has been solved in the infinite domain $0 \le \eta < \infty$. Instead, a finite domain in η direction can be used, with η chosen large enough to ensure that the solutions are not affected by further increasing η. Convergence is achieved only when the absolute value of every unknown for last two approximations differ only by 10^{-5} at all values of η in $0 \le \eta < \eta_\infty$. Uniform step size η = 0.001 is taken. Less than seven approximations are required to satisfy the convergence criteria for all ranges of the parameters studied here.

Results and discussion

In this paper the effect of variable thermal conductivity in the presence of porous medium on the flow and temperature distribution of second grade fluid and second order fluid over an exponentially stretching sheet in the presence of radiation is investigated. The governing equations were developed and transformed using appropriate similarity transformations and than solved numerically using fourth-order Runge-Kutta method with shooting technique. These results are in excellent agreement with the results solved by finite difference method (tabs. 1 and 2). Numerical computations of these equations have been carried out to study the effect of various physical parameters such as viscoelastic parameter k_1^*, variable thermal conductivity parameter ε, radiation parameter Tr, porosity parameter R, and variable Prandtl number Pr are shown graphically in figs. 2-11.

Table 1. Table for second order fluid: Numerical values of wall temperature gradient $-\theta'(0)$ in PST case and wall temperature $\phi(0)$ in PHF case for different values of various physical parameters

k_1^*	Pr	R	Tr	ε	Runge-Kutta solution		Finite difference solution	
					$-\theta'(0)$	$\phi(0)$	$-\theta'(0)$	$\phi(0)$
0.0	3.0	0.5	2.0	–0.1	1.144627	0.970075	1.144619	0.970062
0.2					1.091874	1.018064	1.091846	1.018053
0.3					1.058347	1.051251	1.058293	1.051199
0.5					0.968550	1.152470	0.968547	1.152458
0.2	3.0	0.5	2.0	0.0	1.117246	0.895057	1.117185	0.895044
				–0.1	1.091874	1.018067	1.091846	1.018053
				–0.2	1.063797	1.187684	1.063788	1.187598
				–0.3	1.032549	1.451677	1.032532	1.451659
0.2	3.0	0.5	2.0	–0.1	1.091874	1.018067	1.091846	1.018053
			5.0		0.631447	1.834565	0.631380	1.834548
			7.0		0.495316	2.447824	0.495298	2.447792
			10		0.374963	3.581044	0.374952	3.581031
0.2	3.0	0.0	2.0	–0.1	1.206002	0.919871	1.205999	0.919857
		0.5			1.091874	1.018067	1.091846	1.018053
		1.0			1.004977	1.109038	1.004965	1.109021
0.2	3.0	0.5	2.0	–0.1	1.091874	1.018067	1.091846	1.018053
	4.0				1.338435	0.827095	1.338332	0.827086
	5.0				1.557093	0.709577	1.557071	0.709563
	7.0				1.937313	0.569362	1.937245	0.569347

Figures 2(a) and 2(b) represents the temperatures profile for viscoelastic parameter k_1^* for second grade fluid ($k_1^* < 0$) and figs. 3(a) and 3(b) represents the temperatures profile for viscoelastic parameter k_1^* for second order fluid ($k_1^* > 0$) for PST and PHF cases, respectively, in the presence of radiation, variable thermal conductivity parameter, and porous medium when Pr = 3. From figs. 2(a) and 2(b) it is evident that to increase the magnitude of viscoelastic parameter, $|k_1^*|$, is to decrease the dimensionless temperatures profile for second grade fluid, whereas the opposite trend is observed for second order fluid in figs. 3(a) and 3(b) for both PST and PHF cases. This means that the heat transfer rate from the surface decrease with increasing $|k_1^*|$ for second grade fluid, but for second order fluid the heat transfer will be enhanced as k_1^* increases. Results for temperatures profiles in PST and PHF cases are qualitatively similar.

Table 2. Table for second grade fluid: Numerical values of wall temperature gradient $-\theta'(0)$ in PST case and wall temperature $\phi(0)$ in PHF case for different values of various physical parameters

k_1^*	Pr	R	Tr	ε	Runge-Kutta solution		Finite difference solution	
					$-\theta'(0)$	$\phi(0)$	$-\theta'(0)$	$\phi(0)$
0.0 –0.2 –0.3 –0.5	3.0	0.5	2.0	–0.1	1.144627 1.184419 1.200911 1.228898	0.970075 0.936894 0.923828 0.902505	1.144619 1.184408 1.200902 1.228870	0.970062 0.936873 0.923794 0.902488
–0.2	3.0	0.5	2.0	0.0 –0.1 –0.2 –0.3	1.207590 1.184419 1.158789 1.130282	0.828095 0.936894 1.082933 1.296203	1.207564 1.184408 1.158742 1.130257	0.828071 0.936873 1.082889 1.296192
–0.2	3.0	0.5	2.0 5.0 7.0 10	–0.1	1.184419 0.709931 0.564986 0.433926	0.936894 1.606950 2.082045 2.885222	1.184408 0.709891 0.564976 0.433898	0.936873 1.606937 2.082029 2.885195
–0.2	3.0	0.0 0.5 1.0	2.0	–0.1	1.279464 1.184439 1.108640	0.866474 0.936894 1.002282	1.279442 1.184408 1.108599	0.866461 0.936873 1.002267
–0.2	3.0 4.0 5.0 7.0	0.5	2.0	–0.1	1.184419 1.432919 1.651912 2.031349	0.936894 0.772226 0.668932 0.543291	1.184408 1.432888 1.651896 2.031338	0.936873 0.772197 0.668919 0.543275

The effects of variable thermal conductivity parameter ε on the temperatures profile for second grade fluid ($k_1^* < 0$) are shown in fig. 4(a) and for second order fluid ($k_1^* > 0$) are shown in fig. 5(a) for PST case, in the presence of radiation and porous medium when Pr = 3. From these figures it is clear that increase of variable thermal conductivity parameter ε, also increases the temperatures distribution in PST case for both fluids. Figures 4(b) and 5(b) shows the graphical representation of temperatures profile with distance η for various values of variable thermal conductivity parameter ε on the temperatures profile for second grade fluid ($k_1^* < 0$) and

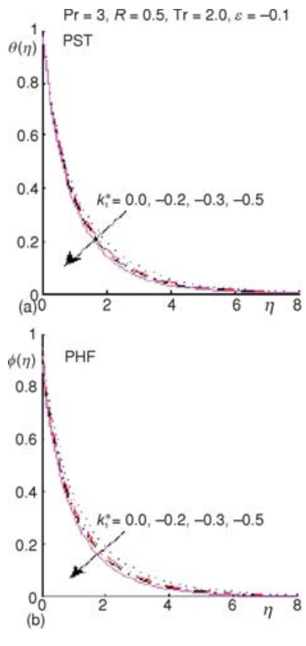

Figure 2. Effects of viscoelastic parameter k_1^* on temperatures profiles for second grade fluid

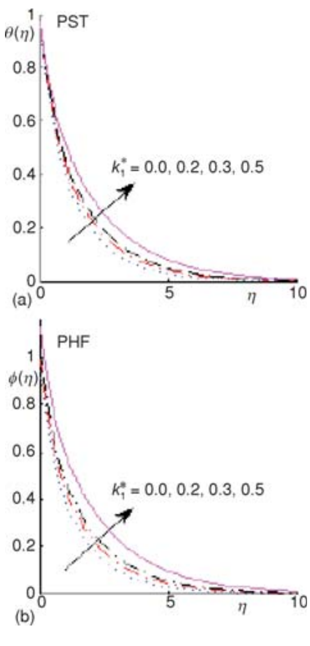

Figure 3. Effects of viscoelastic parameter k_1^* on temperatures profiles for second order fluid

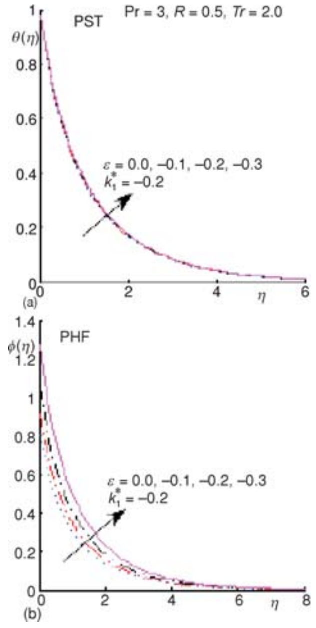

Figure 4. Effects of variable thermal conductivity parameter ε on temperatures profiles for second grade fluid

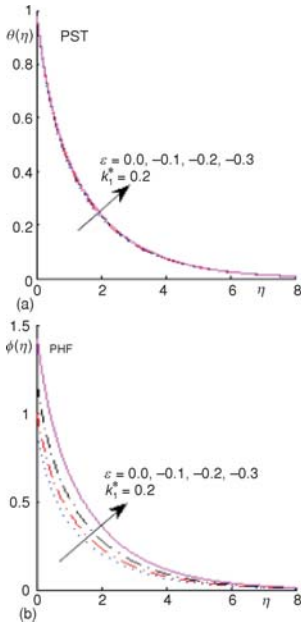

Figure 5. Effects of variable thermal conductivity parameter ε on temperatures profiles for second

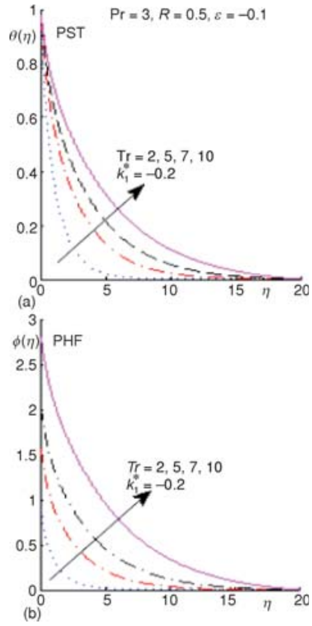

Figure 6. Effects of radiation parameter *Tr* on temperatures profiles for second order fluid

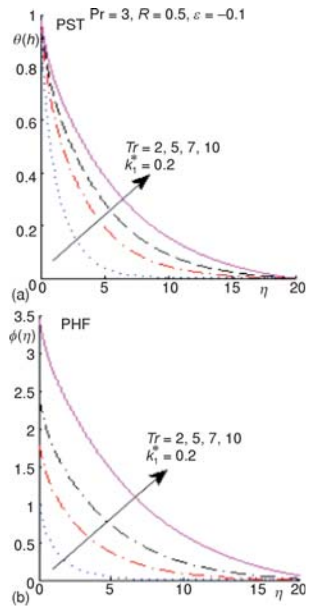

Figure 7. Effects of radiation parameter *Tr* on temperatures profiles for second order fluid

for second order fluid ($k_1^* > 0$) in PHF case. These figures reveal that increase of variable thermal conductivity parameter ε, increases the temperatures distribution for both fluids. It is clear that the nature of the fluids in PHF case is same as in PST case for exponentially stretching sheet but for linearly stretching sheet [1, 2] fluids behave opposite in PST and PHF cases.

Figures 6(a), 6(b), 7(a) and 7(b) illustrates the effects of radiation parameter Tr on temperatures profile for second grade fluid ($k_1^* < 0$) and for second order fluid ($k_1^* > 0$) for PST and PHF cases, in the presence of variable thermal conductivity parameter and porous medium when Pr = 3. Obviously, a significant enhancement in the temperatures profile is observed by increasing the thermal radiation parameter *Tr* for both fluids and in both PST and PHF cases. When the thermal boundary layer thickness is increase in the presence of thermal radiation we observed that the wall temperature gradient decrease, in PST case while the surface temperature increases in PHF case. This result points out that thermal radiation reduces the heat transfer rate from the surface, and thus the radiation should be diminished to have the cooling process at a faster rate.

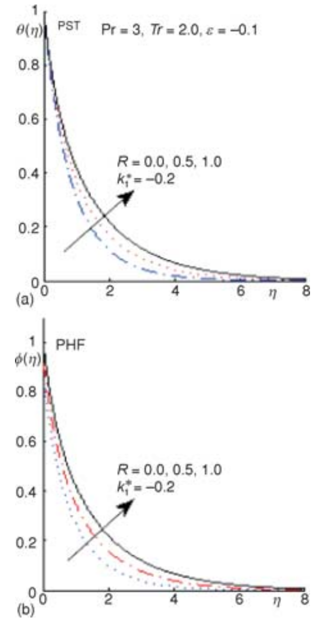

Figure 8. Effects of porosity parameter *R* on temperatures profiles for second grade fluid

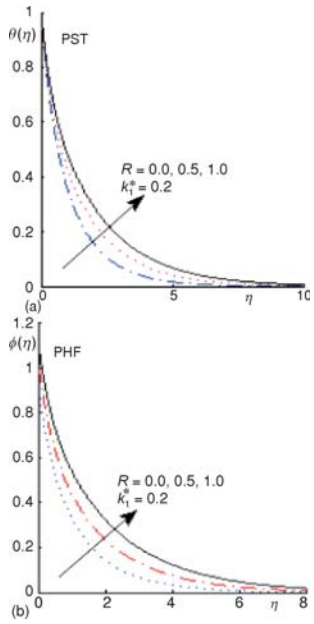

Figure 9. Effects of porosity parameter *R* on temperatures profiles for second order fluid

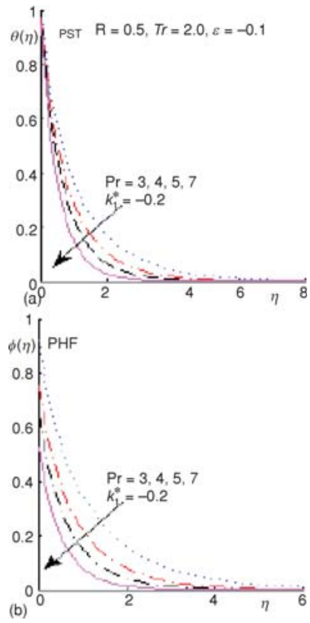

Figure 10. Effects of Prandtl number Pr on temperatures profiles for second grade fluid

Figures 8(a), 8(b), 9(a), and 9(b) shows the effect of porosity parameter R on temperatures profile for second grade fluid ($k_1^* < 0$) and for second order fluid ($k_1^* > 0$) in PST and PHF cases, in the presence of radiation and thermal conductivity parameter when Pr = 3. It is infer from these figures that the temperatures profile increase with an increase in the value of porosity parameter R for both fluids and in PST and PHF cases.

Figures 10(a), 10(b) figs. 11(a), and 11(b) exhibit the temperatures distribution with η for different values of variable Prandtl number Pr in PST and PHF cases for second order fluid, and for second grade fluid respectively, in the presence of radiation, variable thermal conductivity parameter and porous medium. It is apparent from these figures that large values of Prandtl number results in thinning of the thermal boundary layer for both fluids and in both PST and PHF cases. This is in contrast to the effect of other parameter on heat transfer.

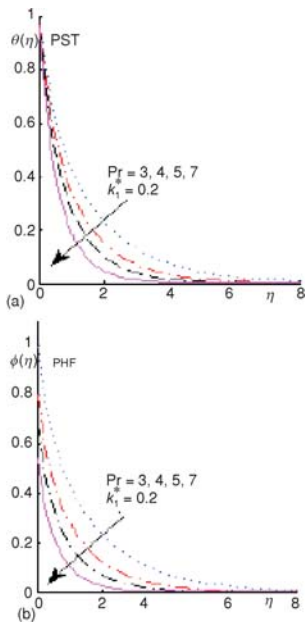

Figure 11. Effects of Prandtl number Pr on temperatures profiles for second order fluid

Conclusions

Important findings of our analysis obtained by the graphical representation are listed as follow:

- Increase in the magnitude of viscoelastic parameter, $|k_1^*|$, decreases the dimensionless temperatures profile for second grade fluid ($k_1^* < 0$) and rise temperatures profile for second order fluid ($k_1^* > 0$) for both PST and PHF cases.
- The variable thermal conductivity also has an impact in enhancing the temperatures profile for both fluids and in both cases (PST and PHF). Hence fluid with less thermal conductivity may be opted for effective cooling.
- The effects of porosity parameter R increases the temperatures profile and hence reduces the heat transfer rate from the surface for both fluids and in both PST and PHF cases. Thus, it may be used to decrease the rate of cooling.
- Radiation should be kept minimum by regulating the temperature of the system for both fluids and in both PST and PHF cases.
- The effect of increasing the values of Prandtl number is to decrease the thermal boundary layer thickness for both fluids and in both PST and PHF cases. Thus, it may be used to increase the rate of cooling.

Acknowledgments

The authors are thankful to the Council of Scientific and Industrial Research, New Delhi, for providing financial support through Grant No. 08/043(0005)/2008-EMR-1. Authors thank the reviewers for their constructive suggestions and comments which have improved the quality of the paper considerably.

References

[1] Sakiadis, B. C., Boundary Layer Behaviour on Continuous Solid Surface: I – Boundary Layer Equations for Two Dimensional and Axisymmetric Flows, *AIChE. J., 7* (1961), 1, pp. 26-28

[2] Sakiadis, B. C., Boundary Layer Behaviour on Continuous Solid Surface: II – Boundary Layer on a Continuous Flat Surface, *AIChE.J., 7* (1961), 2, pp. 221-225

[3] Kumaran,V., Ramanaiah,G., A Note on the Flow over a Stretching Sheet, *Acta Mech., 116* (1996), 1-4, pp. 229-233

[4] Ali, M. E., On Thermal Boundary Layer on a Power Law Stretched Surface with Suction or Injection, *Int. J. Heat Mass Flow, 16* (1995), 4, pp. 280-290

[5] Elbashbeshy, E. M. A., Heat Transfer Over an Exponentially Stretching Continuous Surface with Suction, *Arch. Mech., 53* (2001), 6, pp. 643-651

[6] Magyari, E., Keller, B., Heat and Mass Transfer in the Boundary Layers on an Exponentially Stretching Continuous Surface, *J.Phys. D Appl. Phys.,32* (1999), 5, pp. 577-585

[7] Khan, S. K., Sanjayanand, E., Viscoelastic Boundary Layer Flow and Heat Transfer over an Exponentially Stretching Sheet, *Int. J. Heat Mass Transfer, 48* (2005), 8, pp. 1534-1542

[8] Sanjayanand, E., Khan, S. K., On Heat and Mass Transfer in a Viscoelastic Boundary Layer Flow over an Exponentially Stretching Sheet, *Int. J. Therm. Sci., 45* (2006), 8, pp. 819-828

[9] Andersson, H. I., Note: MHD Flow of a Viscoelastic Fluid Past a Stretching Surface, *Acta Mech., 1-4, 95* (1992), 1-4, pp. 227-230

[10] Vajravelu, K., Flow and Heat Transfer in a Saturated Porous Medium, *ZAMM, 74* (1994), 12, pp. 605-614

[11] Subhas, A., Veena, P., Viscoelastic Fluid Flow and Heat Transfer in a Porous Medium over a Stretching Sheet, *Int. J. Nonlinear Mech. 33* (1998), 3, pp. 531-540

[12] Eldabe Nabil,T. M., Mohamed Mona, A. A., Heat and Mass Transfer in Hydromagnetic Flow of the Non-Newtonian Fluid with Heat Source over an Accelerating Surface through a Porous Medium, *Chaos, Solutions and Fractals, 13* (2002), 4, pp. 907-917

[13] Cogley, A. C., *et al.*, Differential Approximation for Radiation in a Non-Gray Gas Near Equilibrium, *AIAA J, 6* (1968), 3, pp. 551-553

[14] Raptis, A., Radiation and Viscoelastic Flow, *Int. Comm. Heat Mass Transfer, 26* (1999), 6, pp. 889-895

[15] Raptis, A., Technical Note: Flow of a Micropolar Fluid Past Continuously Moving Plate by the Presence of Radiation, *Int. J. Heat Mass Transfer, 41* (1998), 18, pp. 2865-2866

[16] Raptis, A., Perdikis, C., Viscoelastic Flow by the Presence of Radiation, *ZAMM, 78* (1998), 4, pp. 277-279
[17] Siddheshwar, P. G., Mahabaleshwar US, Effect of Radiation and Heat Source on MHD Flow of a Viscoelastic Liquid and Heat Transfer over a Stretching Sheet, *Int. J. Non-Linear Mech., 40* (2005), 6, pp. 807-820
[18] Khan, S. K., Heat Transfer in a Viscoelastic Fluid Flow over a Stretching Surface with Heat Source/Sink, Suction/Blowing and Radiation, *Int. J. Heat and Mass Transfer, 49* (2006), 3, pp. 628-639
[19] Sajid, M., Hayat,T., Influence of Thermal Radiation on the Boundary Layer Flow Due to an Exponentially Stretching Sheet, *Int. Comm. Heat Mass Transfer, 35* (2008), 3, pp. 347-356
[20] Abel, M. S., *et al*., Flow and Heat Transfer in a Power-Law Fluid over a Stretching Sheet with Variable Thermal Conductivity and Non-Uniform Heat Source, *Int. J. Heat and Mass Transfer, 52* (2009), 11, pp. 2902-2913
[21] Chiam,T. C., Heat Transfer in a Fluid with Variable Thermal Conductivity over a Linearly Stretching Sheet, *Acta. Mech., 129* (1998), 1-2, pp. 63-72
[22] Chiam,T. C., Heat Transfer with Variable Conductivity in a Stagnation-Point Flow Towards a Stretching Sheet, *Int. Commun. Heat Mass Transfer, 23* (1996), 2, pp. 239-248
[23] Abel, M. S., Mahesha, N., Heat Transfer in MHD Viscoelastic Fluid Flow over a Stretching Sheet with Variable Thermal Conductivity, Non-Uniform Heat Source and Radiation, *Appl. Math. Modeling, 32* (2008), 10, pp. 1965-1983
[24] Rahman, M. M., Salahuddin, K. M., Study of Hydromagnetic Heat and Mass Transfer Flow over an Inclined Heated Surface with Variable Viscosity and Electric Conductivity, *Commun Nonlinear Sci. Numer. Simulat., 15* (2010), 8, pp. 2073-2085
[25] Chen, H.-C., On the Analytical Solution of MHD Flow and Heat Transfer for Two Types of Viscoelastic Fluid over a Stretching Sheet with Energy Dissipation, Internal Heat Source and Thermal Radiation, *Int. J. Heat Mass Transfer, 53* (2010), 19, pp. 4264-4273
[26] Rivilin, R. S., Ericksen,J. L., Stress Deformation Relations for Isotropic Materials, *J. Ration. Mech. Anal., 4* (1955), 2, pp. 323-425
[27] Coleman, B. D., Noll, W., An Aproximation Theorem for Functionals with Applications in Continuum Mechanics, *Arch Rational Mech. Anal.,* (1960), 6, pp. 355-370
[28] Dunn, J. E., Fosdick, R. L., Thermodynamics Stability and Boundedness of Fluids of Complexity and Fluids of Second Grade, *Arch.Ration.Mech. Anal., 56* (1974), 3, pp. 191-252
[29] Fosdick, R. L., Rajagopal.K.R, Anomalous Features in the Model of Second Order Fluids, *Arch. Ration. Mech. Anal., 70* (1979), 2, pp. 145-152
[30] Beard, D. W., Walters, K., Elastico-Viscous Boundary Layer Flows: Part 1. Two Dimensional Flow Near a Stagnation Point, *Proc. Camb. Phil. Soc.*, (1964), 60, pp. 667-674
[31] Nag, P. K., *Heat and Mass Transfer*, 2nd ed., Tata McGraw-Hill, New Delhi, 2008
[32] Rahman, M. M., *et al*., Heat Transfer in Micropolar Fluid along an Inclined Permeable Plate with Variable Fluid Properties, *I. J. Ther. Sci., 49* (2010), 6, pp. 993-1002

Paper submitted: November 2, 2011
Paper revised: August 24, 2012
Paper accepted: August 24, 2012