ON ANALYTICAL STUDY OF FACTIONAL OLDROYD-B FLOW IN ANNULAR REGION OF TWO TORSIONALLY OSCILLATION CYLINDERS

by

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The velocity field and the associated shear stress corresponding to the torsional oscillatory flow of a fractional Oldroyd-B fluid, also called generalized Oldroyd-B fluid (GOF), between two infinite coaxial circular cylinders, are determined by means of the Laplace and Hankel transforms. Initially, the fluid and cylinders are at rest and after some time both cylinders suddenly begin to oscillate around their common axis with different angular frequencies of their velocities. The exact analytic solutions of the velocity field and associated shear stress, that have been obtained, are presented under integral and series forms in terms of generalized G and R functions. Moreover, these solutions satisfy the governing differential equation and all imposed initial and boundary conditions. The respective solutions for the motion between the cylinders, when one of them is at rest, can be obtained from our general solutions. Furthermore, the corresponding solutions for the similar flow of classical Oldroyd-B, generalized Maxwell, classical Maxwell, generalized second grade, classical second grade and Newtonian fluids are also obtained as limiting cases of our general solutions.

Keywords: Generalized Oldroyd-B fluid, shear stress, torsional oscillatory flow, finite Hankel transform

1. Introduction

Flows in the neighborhood of spinning or oscillating bodies are of interest to both academic workers and industry. Among them, the flows between oscillating cylinders are some of the most important and interesting problems of motion. As early as 1886, Stokes [1] established an exact solution for the rotational oscillations of an infinite rod immersed in a classical linearly viscous fluid. Casarella and Laura [2] obtained an exact solution for the motion of the same fluid due to both longitudinal and torsional oscillations of the rod. Later, Rajagopal [3] found two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal and Bhatnagar [4]. Others interesting results have been recently obtained by Hayat *et al* [5] and Fetecau *et al* [10] and references therein [6 -8].

The non-Newtonian fluids are increasingly being considered more important and appropriate in technological applications than the Newtonian fluids. Strictly speaking, the linear relation between stress and the rate of strain does not exist for a lot of real fluids, such as blood, oils, paints and polymeric solutions. In general, the analysis of the behavior of the fluid motion of the non-Newtonian fluids tends to be much more complicated and subtle in comparison with that of the Newtonian fluids. There have been a fairly large number of flows of Newtonian fluids for which a closed form analytical solution is possible. However, for non-Newtonian fluids such exact solutions are rare.

In order to describe the rheological properties of wide classes of materials more clearly and deeply, the rheological constitutive equations with fractional derivatives have been introduced for a long time, which are discussed in the papers given by Friedrich [25], Bagley [33], Glockle and Nonnenmacher [34], Rossikhin and Shitikova [35], [36], Mainardi [37], Mainardi and Gorenflo [38], Makris and Constantinou [23] and the references therein. The Oldroyd-B model contains as a special case the Maxwell model for which an inadequacy has been pointed out by Choi et al. [39]. In a simple shear flow of a real fluid, it predicts a linear relation between shear rate and shear stress. Furthermore, for the Maxwell model it was not possible to achieve satisfactory fit of experimental data over the entire range of frequencies [23]. A very good fit of experimental data was achieved when the ordinary Maxwell model has been replaced by the Maxwell model with fractional calculus [24]. Recently, the fractional calculus has encountered much success in the description of viscoelasticity. Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of the polymer solutions and melts. In other cases, it has been shown that the constitutive equations employing fractional derivatives are also linked to molecular theories [25]. At least the modified viscoelastic models are appropriate to describe the behavior for Xanthan gum and Sesbania gel [26]. The starting point of the fractional derivative models of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional differential operator. This generalization allows us to define precisely non-integer order integrals or derivatives [13]. There is a vast literature dealing with such fluids, but we shall recall here only a few of the most recent papers [27-32, 40, 41].

In the literature, we can find a lot of work corresponding to the oscillatory flows of viscoelastic non-Newtonian fluids between two cylinders, when one of them is oscillating (see references therein), while the attempts to achieve exact solutions, when both cylinders are oscillating, are scarcely met. As far as the knowledge of authors is concerned, no attempt has been made regarding the torsional oscillatory flow of GOF in the annular region of two infinitely long coaxial circular cylinders, when both of them are oscillating simultaneously. So the aim of this paper is to examine the torsional oscillatory motion of a GOF between two infinite coaxial circular cylinders, both of them oscillating around their common axis with given constant angular frequencies ω_1 and ω_2. Velocity field and associated tangential stress of the motion are determined by using Laplace and Hankel transforms and are presented under integral and series forms in terms of the generalized G and R functions. It is worthy to point out that the solutions that have been obtained satisfy the governing differential equation and all imposed initial and boundary conditions as well. The solutions corresponding to the similar flow of classical Oldroyd-B, generalized Maxwell, classical Maxwell, and Newtonian fluids are also determined as special cases of our general solutions. Unlike other authors, the corresponding solutions for generalized as well as classical second grade fluid are also achieved from general solutions. Furthermore, the respective solutions for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions.

2 Torsional oscillations between two cylinders

2.1 Constitutive equations

The constitutive equations of an incompressible GOF are given by [19, 20]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda \frac{D\mathbf{S}}{Dt} = \mu\left(1 + \lambda_r \frac{D}{Dt}\right)\mathbf{A} \tag{1}$$

Where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ with $\mathbf{L}$ the velocity gradient, λ and λ_r are the material constants and $D\mathbf{S}/Dt$ and $D\mathbf{A}/Dt$ are defined by

$$\frac{D\mathbf{S}}{Dt} = D_t^{\alpha}\mathbf{S} + \mathbf{V}.\nabla\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \qquad \frac{D\mathbf{A}}{Dt} = D_t^{\beta}\mathbf{A} + \mathbf{V}.\nabla\mathbf{A} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T \tag{2a, b}$$

Here $\mathbf{V}$ is the velocity vector, ∇ is the gradient operator, the subscript T denotes the transpose operation and the fractional differential operators D_t^{α} and D_t^{β} are defined as [9, 21, 22]

$$D_t^p = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}d\tau, \qquad 0 < p < 1 \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function. This mode can be reduced to classical Oldroyd-B model when $\alpha \to 1$ and $\beta \to 1$, to the fractional (or generalized) Maxwell model when $\lambda_r \to 0$, to the classical Maxell model when $\lambda_r \to 0$ ad $\alpha \to 1$, and to Newtonian model when $\lambda \to 0$ and $\lambda_r \to 0$.

2.2 Mathematical formulation of problem and governing equation

Suppose that an incompressible GOF is situated in the annular region between two infinite straight circular cylinders of radii R_1 and R_2 $(R_2 > R_1)$. At time $t = 0$, the fluid and cylinders are at rest. At time, $t = 0^+$, inner and outer cylinders suddenly begin to oscillate around their common axis $(r = 0)$ with the velocities $W_1 \sin(\omega_1 t)$ and $W_2 \sin(\omega_2 t)$. Owing to the shear, the fluid between the cylinders is gradually moved, its velocity being of the form

$$\mathbf{v} = \mathbf{v}(r,t) = \upsilon(r,t)e_\theta \tag{4}$$

where e_θ is the unit vector along θ-axis, of the cylindrical coordinate system (r,θ,z). For such flows the constraint of incompressibility is automatically satisfied. Since the velocity field (4) depends only on r and t, so we assume that the extra stress tensor , $\mathbf{S}$ is also independent of θ and z. Furthermore, if the fluid is assumed to be at rest at the moment $t = 0$ then

$$\mathbf{S}(r,0) = 0 \tag{5}$$

Equalities (1b), (4) and (5) imply $S_{rr} = S_{\theta z} = 0$ and relevant equation

$$\left(1+\lambda D_t^{\alpha}\right)\tau\left(r,t\right) = \mu\left(1+\lambda_r D_t^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\upsilon\left(r,t\right) \tag{6}$$

Where $\tau\left(r,t\right) = S_{r\theta}\left(r,t\right)$ is the shear stress, which is different from zero. In the absence of body forces and pressure gradient in the axial direction, the balance of the linear momentum leads to the meaningful equation

$$\rho\frac{\partial\upsilon\left(r,t\right)}{\partial t} = \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau\left(r,t\right) \tag{7}$$

Eliminating $\tau\left(r,t\right)$ between eqs. (6) and (7) we get the governing differential equation of our problem, as follows

$$\left(1+\lambda D_t^{\alpha}\right)\frac{\partial\upsilon\left(r,t\right)}{\partial t} = \nu\left(1+\lambda_r D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\upsilon\left(r,t\right) \ , \quad r\in\left(R_1,R_2\right), \quad t>0 \tag{8}$$

The above governing equation (8) can be reduced to the governing equation (corresponding to the similar flow) of generalized second grade fluid when $\lambda\to 0$ and to the classical second grade fluid when $\lambda\to 0$ and $\beta\to 1$. The appropriate initial and boundary conditions, for the present problem, are

$$\upsilon\left(r,0\right) = \frac{\partial\upsilon\left(r,0\right)}{\partial t} = 0, \ r\in\left(R_1,R_2\right) \tag{9}$$

$$\upsilon\left(R_1,t\right) = W_1\sin\left(\omega_1 t\right) \ , \ \upsilon\left(R_2,t\right) = W_2\sin\left(\omega_2 t\right) \tag{10a, b}$$

To solve this problem, we shall use as in [16, 17], the Laplace and Hankel transforms.

2.3 Calculation of the velocity field

Applying the Laplace transform to Eqs. (8) - (10) and using the Laplace transform formula for sequential fractional derivatives [13], we obtain the ordinary differential equation

$$\frac{\partial^2\bar{\upsilon}\left(r,q\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\upsilon}\left(r,q\right)}{\partial r}-\frac{1}{r^2}\bar{\upsilon}\left(r,q\right)-\frac{q\left(\lambda q^{\alpha}+1\right)}{\nu\left(\lambda_r q^{\beta}+1\right)}\bar{\upsilon}\left(r,q\right) = 0; \ r\in\left(R_1,R_2\right) \tag{11}$$

where the image function $\bar{\upsilon}\left(r,q\right)$ of $\upsilon\left(r,t\right)$ has to satisfy the conditions

$$\bar{\upsilon}\left(R_1,q\right) = \frac{W_1\omega_1}{q^2+\omega_1^2} \ , \ \bar{\upsilon}\left(R_2,q\right) = \frac{W_2\omega_2}{q^2+\omega_2^2} \tag{12a, b}$$

In the following, let us denote by [18]

$$\bar{\upsilon}_n(q)=\int_{R_1}^{R_2} r\bar{\upsilon}(r,q)B_1(rr_n)dr\,,\quad n=1,2,3... \tag{13}$$

the finite Hankel transform of $\bar{\upsilon}(r,q)$, where r_n are the positive roots of the transcendental equation $B_1(R_1 r)=0$ and

$$B_1(rr_n)=J_1(rr_n)Y_1(R_2 r_n)-J_1(R_2 r_n)Y_1(rr_n) \tag{14}$$

In the above relation, $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of order ***one*** of the first and second kind, respectively. Applying the finite Hankel transform (13) to Eq. (11) and taking into account the conditions (12), we find that [14]

$$\frac{2W_2\omega_2}{\pi\left(q^2+\omega_2^2\right)}-\frac{2W_1\omega_1}{\pi\left(q^2+\omega_1^2\right)}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)}-r_n^2\bar{\upsilon}_n(q)-\frac{q\left(\lambda q^{\alpha}+1\right)}{\nu\left(\lambda_r q^{\beta}+1\right)}\bar{\upsilon}_n(q)=0 \tag{15}$$

or equivalently,

$$\bar{\upsilon}(\mathrm{q})=\frac{2\nu W_2\omega_2\left(\lambda_r q^{\beta}+1\right)}{\pi\left(q^2+\omega_2^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}-$$
$$-\frac{2\nu W_1\omega_1\left(\lambda_r q^{\beta}+1\right)}{\pi\left(q^2+\omega_1^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} \tag{16}$$

In order to determine $\bar{\upsilon}(r,q)$ from $\bar{\upsilon}_n(q)$, we firstly write $\bar{\upsilon}_n(q)$ under the suitable form as follows

$$\bar{\upsilon}(q)=\frac{2W_2\omega_2}{\pi r_n^2\left(q^2+\omega_2^2\right)}-\frac{2W_1\omega_1}{\pi r_n^2\left(q^2+\omega_1^2\right)}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)}-$$
$$-\frac{2W_2\omega_2 q\left[\lambda q^{\alpha}+1\right]}{\pi r_n^2\left(q^2+\omega_2^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda q^{\beta}+1\right)\right]}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} \tag{17}$$

and use the inverse Hankel transform formula [14]

$$\bar{\upsilon}(r,q)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_1^2(R_1 r_n)B_1(rr_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\bar{\upsilon}_n(r,q) \tag{18}$$

and (A1) from appendix, we obtain

$$\bar{\upsilon}(r,q)=\frac{W_1R_1\left(R_2^2-r^2\right)\frac{\omega_1}{q^2+\omega_1^2}+W_2R_2\left(r^2-R_1^2\right)\frac{\omega 2}{q^2+\omega_2^2}}{\left(R_2^2-R_1^2\right)r}-\pi\sum_{n=1}^{\infty}\frac{J_1\left(R_1r_n\right)B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}\times$$

$$\times\left[W_2\omega_2J_1\left(R_1r_n\right)\frac{q\left(\lambda q^{\alpha}+1\right)}{\left(q^2+\omega_2^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}-W_1\omega_1J_1\left(R_2r_n\right)\frac{q\left(\lambda q^{\alpha}+1\right)}{\left(q^2+\omega_1^2\right)\left[\lambda q^{\alpha+1}+q\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}\right]$$

(19)

Now, in order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inversion Laplace transform method [11, 12], writing

$$\frac{\lambda q^{\alpha}+1}{\lambda q^{\alpha+1}+q\nu r_n^2\left(\lambda_r q^{\beta}+1\right)}=\sum_{k=0}^{\infty}\sum_{l,m\geq 0}^{l+m=k}\frac{k!\lambda_r^m}{l!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{\beta m-k-1}}{\left(q^{\alpha}+\lambda^{-1}\right)^k} \tag{20}$$

and using (A2), where [15]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{(c)_j\,d^j t^{(j+c)a-b-1}}{j!\Gamma\left[(j+c)a-b\right]} \tag{21}$$

is the generalized G function and $(c)_j$ is the Pochhammer polynomial [15].

Finally, eqs.(19)-(21) and (A5) give the velocity field

$$\upsilon(r,t)=\frac{W_1R_1\left(R_2^2-r^2\right)\sin\left(\omega_1t\right)+W_2R_2\left(r^2-R_1^2\right)\sin\left(\omega_2t\right)}{\left(R_2^2-R_1^2\right)r}-$$

$$-\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l,m\geq 0}^{l+m=k}\frac{k!\lambda_r^m}{l!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{J_1\left(R_1r_n\right)B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}\times \tag{22}$$

$$\times\left[W_2\omega_2J_1\left(R_1r_n\right)\cos\left(\omega_2t\right)-W_1\omega_1J_1\left(R_2r_n\right)\cos\left(\omega_1t\right)\right]*G_{\alpha,\beta m-k-1,k}\left(-\lambda^{-1},t\right)$$

where "$*$" denotes the convolution of two functions.

2.4 Calculation of the shear stress

Applying the Laplace transform to eq. (6), we find that

$$\bar{\tau}(r,q)=\frac{\mu\left(\lambda_r q^{\beta}+1\right)}{\lambda q^{\alpha}+1}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\upsilon}(r,q) \tag{23}$$

where

$$\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\upsilon}(r,q)=\frac{2R_1R_2}{\left(R_2^2-R_1^2\right)r^2}\left(\frac{R_1W_2\omega_2}{q^2+\omega_2^2}-\frac{R_2W_2\omega_1}{q^2+\omega_1^2}\right)+\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left[(1/r)B_1(rr_n)-r_n\tilde{B}_1(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times \tag{24a}$$

$$\times\left[W_2\omega_2J_1(R_1r_n)\frac{q\left(\lambda q^{\alpha}+1\right)}{\left(q^2+\omega_2^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}-\frac{q\left(\lambda q^{\alpha}+1\right)}{\left(q^2+\omega_1^2\right)\left[\lambda q^{\alpha+1}+q+\nu r_n^2\left(\lambda_r q^{\beta}+1\right)\right]}\right]$$

has been obtained from (19) and (A6), where in the above relation

$$\tilde{B}_1(rr_n)=J_0(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_0(rr_n) \tag{24b}$$

Introducing (24) into (23), applying again the discrete inversion Laplace transform to the obtained result and using (A3) and (A5), where

$$R_{a,b}(c,d,t)=\sum_{j=0}^{\infty}\frac{c^j\,(t-d)^{(j+1)a-b-1}}{\Gamma\left[(j+1)a-b\right]} \tag{25}$$

is R function [15], we find for the shear stress the expression

$$\tau(r,t)=\frac{2\mu R_1R_2}{\lambda\left(R_2^2-R_1^2\right)r^2}\begin{bmatrix}R_1W_2\left\{\sin(\omega_2t)+\omega_2\lambda_rR_{2,\beta}\left(-\omega_2^2,0,t\right)\right\}-\\ R_2W_1\left\{\sin(\omega_1t)+\omega_1\lambda_rR_{2,\beta}\left(-\omega_1^2,0,t\right)\right\}\end{bmatrix}*R_{\alpha,0}\left(-\lambda^{-1},0,t\right)+$$

$$+\frac{\mu\pi}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l,m\geq0}^{l+m=k+1}\frac{(k+1)!\lambda_r^m}{l!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{J_1(R_1r_n)\left[(1/r)B_1(rr_n)-r_n\tilde{B}_1(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times \tag{26}$$

$$\times\begin{bmatrix}W_2\omega_2J_1(R_1r_n)\left\{G_{\alpha,\beta m-k-2,k+1}\left(-\lambda^{-1},t\right)-\omega_2\sin(\omega_2t)*G_{\alpha,\beta m-k-2,k+1}\left(-\lambda^{-1},t\right)\right\}-\\ -W_1\omega_1J_1(R_2r_n)\left\{G_{\alpha,\beta m-k-2,k+1}\left(-\lambda^{-1},t\right)-\omega_1\sin(\omega_1t)*G_{\alpha,\beta m-k-2,k+1}\left(-\lambda^{-1},t\right)\right\}\end{bmatrix}$$

Making the limits as $\alpha\to1$, $\beta\to1$ into Eqs. (22) and (26), we can recover the corresponding solutions for the classical Oldroyd-B fluid.

3 Limiting case

3.1 Generalized Maxwell fluid

Making $\lambda_r\to0$ into eqs. (22) and (26) and using (A4), we obtain the velocity field

$$\upsilon_{GM}(r,t)=\frac{W_1R_1\left(R_2^2-r^2\right)\sin\left(\omega_1 t\right)+W_2R_2\left(r^2-R_1^2\right)\sin\left(\omega_2 t\right)}{\left(R_2^2-R_1^2\right)r}-\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{J_1\left(R_1r_n\right)B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}\times$$

$$\times\left[W_2\omega_2J_1\left(R_1r_n\right)\cos\left(\omega_2t\right)-W_1\omega_1J_1\left(R_2r_n\right)\cos\left(\omega_1t\right)\right]*G_{\alpha,-k-1,k}\left(-\lambda^{-1},t\right) \tag{27}$$

and associated shear stress

$$\tau_{GM}(r,t)=\frac{2\mu R_1R_2}{\lambda\left(R_2^2-R_1^2\right)r^2}\left[R_1W_2\sin\left(\omega_2t\right)-R_2W_1\sin\left(\omega_1t\right)\right]*R_{\alpha,0}\left(-\lambda^{-1},0,t\right)+$$

$$+\frac{\mu\pi}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{J_1\left(R_1r_n\right)\left[\left(1/r\right)B_1\left(rr_n\right)-r_n\tilde{B}_1\left(rr_n\right)\right]}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}\times \tag{28}$$

$$\times\left[\begin{array}{l}W_2\omega_2J_1\left(R_1r_n\right)\left\{G_{\alpha,-k-2,k+1}\left(-\lambda^{-1},t\right)-\omega_2\sin\left(\omega_2t\right)*G_{\alpha,-k-2,k+1}\left(-\lambda^{-1},t\right)\right\}-\\ \\ -W_1\omega_1J_1\left(R_2r_n\right)\left\{G_{\alpha,-k-2,k+1}\left(-\lambda^{-1},t\right)\omega_1\sin\left(\omega_1t\right)*G_{\alpha,-k-2,k+1}\left(-\lambda^{-1},t\right)\right\}\end{array}\right]$$

corresponding to the Generalized Maxwell fluid, performing the same motion. It is remarkable here that if we have $\alpha\to 1$ in Eqs. (27) and (28), then corresponding solutions for ordinary or classical Maxwell fluid are recovered.

3.2 Generalized second grade fluid

Now, making the limit as $\lambda\to 0$ into Eqs. (22) and (26), we obtain the corresponding solutions for the generalized second grade fluid, which are given by

$$\upsilon_{GSG}(r,t)=\frac{W_1R_1\left(R_2^2-r^2\right)\sin\left(\omega_1t\right)+W_2\,R_2\left(r^2-R_1^2\right)\sin\left(\omega_2t\right)}{\left(R_2^2-R_1^2\right)r}-$$

$$-\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l,m\geq 0}^{l+m=k}\frac{k!\lambda_r^m}{l!m!}\frac{\left(-\nu r_n^2\right)^k}{\Gamma\left(-\beta m+k+1\right)}\frac{J_1\left(R_1r_n\right)B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}\times \tag{29}$$

$$\times\left[W_2\omega_2J_1\left(R_1r_n\right)\cos\left(\omega_2t\right)-W_1\omega_1J_1\left(R_2r_n\right)\cos\left(\omega_1t\right)\right]*t^{-\beta m+k}$$

and

$$\tau_{GSG}(r,t)=\frac{2\mu R_1R_2}{\left(R_2^2-R_1^2\right)r^2}\left[R_1W_2\left\{\sin(\omega_2 t)+\omega_2\lambda_r R_{2,\beta}\left(-\omega_2^2,0,t\right)\right\}-R_2W_1\left\{\sin(\omega_1 t)+\omega_1\lambda_r R_{2,\beta}\left(-\omega_1^2,0,t\right)\right\}\right]+$$

$$+\mu\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l,m\geq 0}^{l+m=k+1}\frac{(k+1)!\lambda_r^m}{l!m!}\frac{\left(-\nu r_n^2\right)^k}{\Gamma(-\beta m+k+2)}\frac{J_1(R_1r_n)\left[(1/r)B_1(rr_n)-r_n\tilde{B}_1(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times$$

$$\times\left[W_2\omega_2J_1(R_1r_n)\left\{t^{-\beta m+k+1}-\omega_2\sin(\omega_2 t)*t^{-\beta m+k+1}\right\}-W_1\omega_1J_1(R_2r_n)\left\{t^{-\beta m+k+1}-\omega_1\sin(\omega_1 t)*t^{-\beta m+k+1}\right\}\right] \tag{30}$$

By making $\beta\to 1$ into (29) and (30), we get the corresponding solutions for the similar flow of classical or ordinary second grade fluid.

3.3 Newtonian fluid

Making the limit as $\alpha\to 1,\ \beta\to 1,\ \lambda\to 1,\ \lambda_r\to 1$ into eqs. (22) and (26), we obtain the corresponding solutions for the Newtonian fluid, which are given by

$$\upsilon_N(r,t)=\frac{W_1R_1\left(R_2^2-r^2\right)\sin(\omega_1 t)+W_2R_2\left(r^2-R_1^2\right)\sin(\omega_2 t)}{\left(R_2^2-R_1^2\right)r}-\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1(R_2r_n)}\times$$

$$\times\begin{bmatrix}\dfrac{W_2\omega_2J_1(R_1r_n)}{\nu^2r_n^4+\omega_2^2}\left\{\nu r_n^2\left(\cos(\omega_2 t)-\exp\left(-\nu r_n^2t\right)\right)+\omega_2\sin(\omega_2 t)\right\}-\\[2ex] -\dfrac{W_1\omega_1J_1(R_2r_n)}{\nu^2r_n^4+\omega_1^2}\left\{\nu r_n^2\left(\cos(\omega_1 t)-\exp\left(-\nu r_n^2t\right)\right)+\omega_1\sin(\omega_1 t)\right\}\end{bmatrix} \tag{31}$$

and

$$\tau_N(r,t)=\frac{2\mu R_1R_2}{\left(R_2^2-R_1^2\right)r^2}\left(R_1W_2\ \sin(\omega_2 t)-R_2W_1\sin(\omega_1 t)\right)+\mu\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left[(2/r)B_1(rr_n)-r_n\tilde{B}_1(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times$$

$$\times\begin{bmatrix}\dfrac{W_2\omega_2J_1(R_1r_n)}{\nu r_n^4+\omega_2^2}\left\{\nu r_n^2\left(\cos(\omega_2 t)-\exp\left(-\nu r_n^2t\right)\right)+\omega_2\sin(\omega_2 t)\right\}-\\[2ex] -\dfrac{W_1\omega_1J_1(R_2r_n)}{\nu r_n^4+\omega_1^2}\left\{\nu r_n^2\left(\cos(\omega_1 t)-\exp\left(-\nu r_n^2t\right)\right)+\omega_1\sin(\omega_1 t)\right\}\end{bmatrix} \tag{32}$$

It is important to point out that the terms containing $exp(\cdot)$ in Eqs. (31) and (32), correspond to the transient parts of the solutions of $\upsilon_N(r,t)$ and $\tau_N(r,t)$, respectively. For large values of time, these terms tend to zero and we remain with the steady-state solutions, which are periodic in time and are independent of the initial condition.

Concluding remarks and results

Our purpose in this paper was to establish exact solutions for the velocity field and shear stress corresponding to the flow of a GOF between to infinite coaxial circular cylinders, by using Laplace and Hankel transforms. The motion of fluid was due to the simple harmonic sine oscillations of both cylinders around their common axis, with different angular frequencies ω_1 and ω_2 of their velocities. It is important to point out that the velocity field and the shear stress for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions by making $W_1 = 0$, $W_2 = W$ and $\omega_2 = \omega$ (when inner cylinder is at rest) or $W_1 = W$, $W_2 = 0$ and $\omega_1 = \omega$ (when outer cylinder is at rest). For instance, the velocity field for the flow of generalized Maxwell fluid, when inner cylinder is at rest and outer cylinder is oscillating, is given by (from eq. (20))

$$\upsilon(r,t) = \frac{WR_2\left(r^2 - R_1^2\right)\sin(\omega t)}{\left(R_2^2 - R_1^2\right)r} - \pi W\omega \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l,m\geq 0}^{l+m=k} \frac{k!\lambda_r^m}{l!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times$$
$$\times \frac{J_1^2(R_1 r_n)B1(r r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}\cos(\omega t) * G_{\alpha,\beta m-k-1,k}\left(-\lambda^{-1}, t\right) \tag{33}$$

The solutions that have been obtained, presented under integral and series forms in terms of the generalized G and R functions, satisfy the governing equation and all imposed initial and boundary conditions. The corresponding solutions for the similar flow of generalized as well as the classical Maxwell and second grade fluids are also achieved as special limiting cases of our general solutions.

Appendix

Some results used in the text:

The finite Hankel transform of the function

$$a(r) = \frac{AR_1\left(R_2^2 - r^2\right) + BR_2\left(r^2 - R_1^2\right)}{\left(R_2^2 - R_1^2\right)r} \tag{A1a}$$

satisfying $a(R_1) = A$ and $a(R_2) = B$ is

$$a_n = \int_{R_1}^{R_2} r a(r) B_1(r r_n) dr = \frac{2B}{\pi r_n^2} - \frac{2A}{\pi r_n^2}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} \tag{A1b}$$

$$L^{-1}\left\{\frac{q^b}{\left(q^a - d\right)^c}\right\} = G_{a,b,c}(d,t);\; Re(ac-b) > 0,\; Re(q) > 0,\; \left|\frac{d}{q^a}\right| < 1 \tag{A2}$$

$$L^{-1}\left\{\frac{e^{dq} q^b}{q^{a-c}}\right\} = R_{a,b}(c,d,t) = \sum_{j=0}^{\infty} \frac{c^j (t-d)^{(j+1)a-b-1}}{\Gamma\left[(j+1)a-b\right]},\; d \ge 0,\; Re\left[(j+1)a - b\right] > 0,\; Re(q) > 0 \tag{A3}$$

$$R_{1,0}(a,0,t) = exp(at) \tag{A4}$$

$$\text{If } u_1(t) = L^{-1}\left\{\bar{u}_1(q)\right\} \text{ and } u_2(t) = L^{-1}\left\{\bar{u}_2(q)\right\} \text{ then} \tag{A5a}$$

$$L^{-1}\left\{\bar{u}_1(q)\bar{u}_2(q)\right\} = (u_1 * u_2)(t) = \int_0^t u_1(t-s)u_2(s)ds = \int_0^t u_1(s)u_2(t-s)ds \tag{A5b}$$

$$\frac{d}{dr}\left[B_1(r r_n)\right] = r_n\left[J_0(r r_n)Y_1(R_2 r_n) - J_1(R_2 r_n)Y_0(r r_n)\right] - \frac{1}{r}B_1(r r_n) \tag{A6}$$

Nomenclature

R_1 -radius of inner cylinder, [m]
R_2 -radius of outer cylinder, [m]
r -radial coordinate, [m]
t -time, [s]

Greek letters

α -fractional order (dimensionless)
β -fractional order (dimensionless)
ρ -fluid density , [$kg.m^{-3}$]
μ - dynamic viscosity [$N.m^{-2}.s$]
$\nu = \mu/\rho$ fluid kinematic viscosity, [$m^2 s^{-1}$]
ω - angular frequency of velocity, [s^{-1}]
τ -shear stress, [$N.m^{-2}$]

Subscripts

1-inner cylinder
2- outer cylinder
GSC- generalized second grade
N -Newtonian

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