CONVERTING FRACTIONAL DIFFERENTIAL EQUATIONS INTO PARTIAL DIFFERENTIAL EQUATIONS

by

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A transform is suggested in this paper to convert fractional differential equations with the modified Riemann-Liouville derivative into partial differential equations, and it is concluded that the fractional order in fractional differential equations is equivalent to the fractal dimension.

Key words: *Modified Riemann-Liouville Derivative, Time-fractional Heat Conduction Equation, Fractional KdV equation*

1. Introduction

In our previous work[1,2], we suggested a fractional complex transform to convert fractional differential equations directly into ordinary differential equations. Though such transform makes solution process extremely simple, it is valid only for a general "wave" solution where the variables *t, x, y,* and *z* can not change freely but follow the following constraint

$$\xi=\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{ky^{\gamma}}{\Gamma(1+\gamma)}+\frac{lz^{\lambda}}{\Gamma(1+\lambda)} \tag{1}$$

where p, q, k and l are constants.

In this paper we will suggest a similar transform to convert fractional differential equations, instead of ordinary differential equations, into partial differential equations.

2. How to convert fractional differential equations to partial differential equations?

Consider the following general fractional differential equation

$$f\left(u,u_t^{(\alpha)},u_x^{(\beta)},u_y^{(\gamma)},u_z^{(\lambda)},u_t^{(2\alpha)},u_x^{(2\beta)},u_y^{(2\gamma)},u_z^{(2\lambda)},\cdots\right)=0, \tag{2}$$

where $u_t^{(\alpha)} = \partial^{\alpha}u(x,y,z,t)/\partial t^{\alpha}$ denotes the modified Riemann-Liouville derivative[3-5]. $0<\alpha\le 1,\ 0<\beta\le 1,\ 0<\gamma\le 1,\ 0<\lambda\le 1$.

We introduce the following transforms

$$s=\frac{qt^{\alpha}}{\Gamma(1+\alpha)} \tag{3}$$

$$X=\frac{px^{\beta}}{\Gamma(1+\beta)} \tag{4}$$

$$Y=\frac{ky^{\gamma}}{\Gamma(1+\gamma)} \tag{5}$$

$$Z=\frac{lz^{\lambda}}{\Gamma(1+\lambda)} \tag{6}$$

where p, q, k and l are constants.

Using the above transforms, we can convert fractional derivatives into classical derivatives:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=q\frac{\partial u}{\partial s} \tag{7}$$

$$\frac{\partial^{\beta}u}{\partial x^{\beta}}=p\frac{\partial u}{\partial X} \tag{8}$$

$$\frac{\partial^{\gamma}u}{\partial y^{\gamma}}=k\frac{\partial u}{\partial Y} \tag{9}$$

$$\frac{\partial^{\lambda}u}{\partial z^{\lambda}}=l\frac{\partial u}{\partial Z} \tag{10}$$

Please note Eqs.(7-10) are valid for only Jumarie's modification of Riemann-Liouville Derivative[3-5].

We can, therefore, easily convert fractional differential equations into partial differential equations, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty.

3. Examples

Consider the following fractional-time heat conduction equation :

$$\frac{\partial^{\alpha}T}{\partial t^{\alpha}}=D\frac{\partial^{2}T}{\partial x^{2}},\ x\in(0,\infty),\ t>0.\quad 0<\alpha\le 1. \tag{11}$$

Using Eq.(3), we have the following partial differential equation

$$q\frac{\partial T}{\partial s}=D\frac{\partial^{2}T}{\partial x^{2}},\ x\in(0,\infty),\ t>0. \tag{12}$$

Now consider the following fractional KdV equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+6u\frac{\partial^{\beta}u}{\partial x^{\beta}}+\frac{\partial^{3\beta}u}{\partial x^{3\beta}}=0. \tag{13}$$

Using the above transformation, we have

$$q\frac{\partial u}{\partial s}+6pu\frac{\partial u}{\partial X}+p^{3}\frac{\partial^{3}u}{\partial X^{3}}=0 \tag{14}$$

For simplicity, we set $p=q=1$, this results in

$$\frac{\partial u}{\partial s}+6u\frac{\partial u}{\partial X}+\frac{\partial^{3}u}{\partial X^{3}}=0 \tag{15}$$

Its soliton solution reads

$$u=\frac{A}{2}\operatorname{sech}^{2}\left\{\frac{\sqrt{A}}{2}(X-As)\right\} \tag{16}$$

or

$$u(x,t)=\frac{A}{2}\mathrm{sech}^2\left\{\frac{\sqrt{A}}{2}\left(\frac{x^{\beta}}{\Gamma(1+\beta)}-\frac{At^{\alpha}}{\Gamma(1+\alpha)}\right)\right\} \qquad (17)$$

4. Discussion and Conclusions

The transformation is very similar to that given in the fractal derivative[6,7], which is defined as

$$\frac{Du(x)}{Dx^{\alpha}}=\lim_{\Delta x\to L_0}\frac{u(A)-u(B)}{\text{The distance between two points A and B}}=\frac{du}{ds}=\lim_{\Delta x\to L_0}\frac{u(A)-u(B)}{kL_0^{\alpha}} \qquad (18)$$

where k is a constant, α is the fractal dimension, the distance between two points in a discontinuous space can be expressed as

$$ds=kL_0^{\alpha} \qquad (19)$$

Comparing Eq.(4) and Eq.(19), we conclude that the fractional order in fractional differential equations is equivalent to the fractal dimension.

The transformation is simple and straightforward, it is extremely accessible to nonmathematicians. The use of the present method requires no special knowledge of fractional calculus.

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